13. Resolution of singularities III

Let us now turn to a detailed explanation of Hironaka's proof of embedded resolution. In large part we follow the proof of Włodarczyk, which consists of a considerable simplification of Hironaka's original proof.

Theorem 13.1 (Embedded resolution)**.** *Let $X = X_0 \subset M = M_0$ be quasi-projective varieties over a field of characteristic zero, where M is smooth.*

Then there is a sequence of blow ups $\sigma_i \colon M_{i+1} \longrightarrow M_i$ of smooth centres, $1 \leq i \leq r-1$, such that if $\Pi \colon N = M_r \longrightarrow M$ denotes the composition, and X_i denotes the strict transform of X, then we have

(1) *$Y = X_r$ is smooth,*
(2) *the induced birational maps $\tau_i \colon X_{i+1} \longrightarrow X_i$ don't depend on the embedding of X into M, and*
(3) *the maps $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$ commute with smooth base change and with any automorphism, not even necessarily over the groundfield.*

Let me comment briefly on the significance of the last statement. Suppose that the groundfield k is not algebraically closed. Let G be the Galois group of the algebraic closure. Let $X' \subset M'$ be what you get by base changing to the algebraic closure. Since Π commutes with the action of G, it follows that resolving $X' \subset M'$ automatically resolves $X \subset M$. So we may safely assume that the groundfield is algebraically closed. So we may assume that the groundfield is $\mathbb{C}$.

Suppose that $X \subset M$ is smooth to begin with. Then it is clear that $r = 0$ and Π is the identity. Let $U \subset X$ be the smooth locus of X. Then U is an open subset of X and we may pick $V \subset M$ open such that $U = X \cap V \subset V$. The map $V \longrightarrow M$ induces a smooth base change. By what we just said, it follows that Π is an isomorphism outside of the singular locus of X.

Unfortunately it is not true that one can find Π which commutes with any base change. For example, suppose that S is a surface with an A_1-singularity. At least locally, S is a quotient singularity,

$$f \colon \mathbb{C}^2 \longrightarrow S = \mathbb{C}^2/\mathbb{Z}_2.$$

But we have already seen that Π does not change a smooth variety, so Π cannot commute with f.

In fact Π does not even commute with taking products. Consider the example of an A_1-singularity

$$S = (x^2 + y^2 + z^2 = 0) \subset \mathbb{C}^3.$$

Then

$$\Pi \colon N = \operatorname{Bl}_p \mathbb{C}^3 \longrightarrow \mathbb{C}^3,$$

blows up the origin. But if

$$X = S \times S \subset \mathbb{C}^6,$$

then the singular locus of X is $\{p\} \times S \cup S \times \{p\}$ and the product morphism would try to blow up this locus, which is not even smooth.

One of Hironaka's great ideas is to write down a smooth hypersurface $H \subset M$ and to construct Π inductively by considering $X \cap H \subset H$. If X is a hypersurface itself, which is locally given by a Weirstrass polynomial

$$y^\mu + f_{\mu-2} y^{\mu-2} + \cdots + f_0 = 0,$$

then we can take $H = (y = 0)$. The idea is that no matter how many times we blow up, as long as the multiplicity of the strict transform of X is exactly μ, then the singular locus of the strict transform of X is contained in the strict transform of H. For this reason, H is called a **hypersurface of maximal contact**.

Suppose that $M = \mathbb{C}^2$ and $X = (y^2 + x^{n+1} = 0) \subset \mathbb{C}^2$. In this case we take $H = (y = 0)$, so that $X \cap H \subset H$ is a point. It is clear that we need to keep track of the scheme structure; $z = X \cap H \subset M$ is a zero dimensional scheme, of length $n+1$. Since we have to keep track of the scheme structure, it is in fact better to work with ideal sheaves $\mathcal{I}$ on M and not just subvarieties X. Notice also that, by way of induction, one has to allow the rather silly possibility that we blow up a divisor on H.

Definition 13.2. *Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth variety M. The **multiplicity** of $\mathcal{I}$ at a point $p \in M$ is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$, where $\mathfrak{m} \subset \mathcal{O}_{M,p}$ is the maximal ideal of $p \in M$.*

*Let $V \subset M$ be a smooth subvariety and let $\sigma \colon N \longrightarrow M$ be the blow up of M along V. The **strict transform** of $\mathcal{I}$ is the ideal sheaf $\mathcal{J} = \sigma^{-1}\mathcal{I} \underset{\mathcal{O}_N}{\otimes} \mathcal{O}_N(-\mu E)$*

Theorem 13.3 (Principalisation of ideals)**.** *Let $\mathcal{I} = \mathcal{I}_0 \subset M = M_0$ be a coherent sheaf of ideal on a quasi-projective variety over a field of characteristic zero, where M is smooth.*

Then there is a sequence of blow ups $\sigma_i \colon M_{i+1} \longrightarrow M_i$ of smooth centres, $1 \leq i \leq r-1$, such that if $\Pi \colon N = M_r \longrightarrow M$ denotes the composition, and $\mathcal{I}_{i+1}$ denotes the strict transform of $\mathcal{I}_i$, then we have

(1) *$\mathcal{I}_r = \mathcal{O}_Y$ is the trivial ideal sheaf,*
(2) *the induced birational maps $\tau_i \colon X_{i+1} \longrightarrow X_i$ don't depend on the embedding of X into M, and*

(3) *the maps $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$ commute with smooth base change and with any automorphism, not even necessarily over the ground-field.*

Lemma 13.4. (13.3) *implies* (13.1).

Proof. Let $\mathcal{I}$ be the ideal sheaf of X. At some point we must blow up the strict transform Y_i of X, either to make Y_i into a divisor, or to cancel off some multiple of the divisor Y_i. Either way, since we only blow up smooth subvarieties of M, Y_i must be smooth. $\square$

Note that we have to take account of the fact that when we restrict to a hypersurface of maximal contact, then we may not be able to cancel off all of the exceptional divisor.

Definition 13.5. *Let $\mathcal{I}$ be a coherent sheaf of ideals and let m be a positive integer. The pair $(\mathcal{I}, m)$ is called a* ***marked ideal****.*

The ***support*** *of a marked ideal $(\mathcal{I}, m)$ is the closed subset where the multiplicity of $\mathcal{I}$ is at least m.*

Let $V \subset M$ be a smooth subvariety and let $\sigma \colon N \longrightarrow M$ be the blow up of M along V. The ***strict transform*** *of $(\mathcal{I}, m)$ is the pair $(\mathcal{J}, m)$, where $\mathcal{J} = \sigma^{-1}\mathcal{I} \underset{\mathcal{O}_N}{\otimes} \mathcal{O}_N(-mE)$.*

We will always suppose that the multiplicity of $\mathcal{J}$ along V is at least m.

Theorem 13.6 (Principalisation of marked ideals)**.** *Let $\mathcal{I} = \mathcal{I}_0 \subset M = M_0$ be a coherent sheaf of ideals on a quasi-projective variety over a field of characteristic zero, where M is smooth. Let m be a positive integer.*

Then there is a sequence of blow ups $\sigma_i \colon M_{i+1} \longrightarrow M_i$ of smooth centres, $1 \leq i \leq r-1$, such that if $\Pi \colon N = M_r \longrightarrow M$ denotes the composition, and $(\mathcal{I}_{i+1}, m)$ denotes the strict transform of $(\mathcal{I}_i, m)$, then we have

(1) *the support of $\mathcal{I}_r$ is empty,*
(2) *the induced birational maps $\tau_i \colon X_{i+1} \longrightarrow X_i$ don't depend on the embedding of X into M, and*
(3) *the maps $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$ commute with smooth base change and with any automorphism, not even necessarily over the ground-field.*

The scheme of the induction is then to show that (13.6) in dimension $n-1$ implies (13.3) in dimension n and that (13.3) in dimension n implies (13.6) in dimension n. The second implication is relatively easy and the first takes some work.

In practice we should also keep track of the exceptional divisors and we should order them. The only thing we really have to worry about is making sure that we never blow up a centre which destroys the fact that these divisors have global normal crossings.

Given the ideal $\mathcal{I}$, how should we define the hypersurface of maximal contact? The answer is relatively easy, just differentiate enough times; if the multiplicity of $\mathcal{I}$ is μ, then the ideal

$$D(I),$$

of all partial derivatives, has multiplicity $\mu - 1$. So pick a general element of

$$f \in D^{\mu-1}(I),$$

and let $H = (f = 0)$. With this choice of a hypersurface of maximal contact, any sequence of blow ups for $\mathcal{I}$ of order m induces a unique choice of blow ups for $(\mathcal{I}|_H, m)$.

There is one more example to illustrate one more complication. Consider

$$X = (x^2 + y^3 - z^6) \subset \mathbb{C}^3.$$

Suppose we pick $H = (x - z^3 = 0)$ as a hypersurface of maximal contact. As

$$x^2 - z^6 + y^3 = (x - z^3)(z + z^3) + y^3,$$

the restriction $Y = X \cap H \subset \mathbb{C}^2 = H$ is a triple line. The original surface has an isolated singularity, so if we are not careful we might have to do more work to resolve $Y \subset H$.

Definition 13.7. *Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth quasi-projective variety. We say that $\mathcal{I}$ is D-**balanced** if*

$$(D^i(\mathcal{I}))^\mu \subset \mathcal{I}^{\mu-i},$$

for all $i < \mu$, where μ is the maximal multiplicity.

If I is D-balanced and H is a hypersurface of maximal contact, then any sequence of blow ups for $\mathcal{I}$ of order m is the same as a sequence of blow ups for $(\mathcal{I}|_H, m)$.

Example 13.8. *Suppose we start with*

$$I = \langle xy - z^n \rangle \subset \mathbb{C}[x, y, z].$$

If we restrict to $x = 0$, we get an n-fold line, which is not correct. Now consider

$$I + D(I)^2 = \langle xy, x^2, y^2, xz^{n-1}, yz^{n-1}, z^n \rangle.$$

If we restrict to $x = 0$, then we get the ideal

$$\langle y^2, yz^{n-1}, z^n \rangle.$$

It is easy to check that resolving this ideal induces the correct resolution of I.

The final problem is that we need to make sure that the resolution process does not depend on our choice of an element $D^{\mu-1}(I)$, so that we can pass from the local picture to the global picture. Recall that it is enough to make sure that the sequence of blow ups commutes with all automorphisms.

Definition 13.9. *We say that $\mathcal{I}$ is* ***symmetric*** *if*

$$D^{\mu-1}(\mathcal{I}) \cdot \mathcal{I} \subset \mathcal{I}.$$

Note that this is very similar to the D-balanced condition. One can show that if $\mathcal{I}$ is symmetric and D-balanced, then we are done. The idea is to change $\mathcal{I}$, to another ideal sheaf $\mathcal{J}$, so that resolving $\mathcal{J}$ induces a resolution of $\mathcal{J}$, where now $\mathcal{J}$ is symmetric and D-balanced.

The problem is that we need to make $\mathcal{J}$ have very large multiplicity. Suppose that $\mathcal{I}$ has maximal multiplicity μ.

$$W(\mathcal{I}) = \{ \prod_{j=0}^{\mu} D^j(\mathcal{I})^{c_j} \mid \sum(\mu - c_j) \geq \mu! \}.$$

Theorem 13.10. *Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth quasi-projective variety M of maximal multiplicity μ. Then*

(1) *$W(\mathcal{I})$ has maximal multiplicity $\mu!$.*
(2) *$W(\mathcal{I})$ is D-balanced.*
(3) *$W(\mathcal{I})$ is symmetric.*
(4) *Blows up for $(\mathcal{I}, \mu)$ are the same as blow ups for $(W(\mathcal{I}), \mu!)$.*

Let us consider how to resolve a threefold $(f = 0)$ of multiplicity 3 in $\mathbb{C}^4$. The first step is to replace

$$I = \langle f \rangle \qquad \text{by} \qquad W(I),$$

which increases the multiplicity from 3 to $6 = 3!$. The next step is to restrict to a hypersurface of maximal contact. Then we repeat the same operation, replacing 6 by $720 = 6!$. It is clear that we need quite a bit more work to implement resolution of singularities on a computer.