HWK #7, DUE WEDNESDAY 3/30

1. Let K be a field. Consider the following property $P(K)$ of K. If $f\colon K^2 \longrightarrow K$ is any function whose restriction to every horizontal and vertical line (that is $K \times \{b\}$ and $\{a\} \times K$) is a polynomial, then f is a polynomial.
(i) Show that $P(\mathbb{C})$ holds (*Hint: observe that the degree is constant on most lines from one family*).
(ii) Show that $P(\overline{\mathbb{Q}})$ fails (*Hint: order the horizontal and vertical lines (separately) and consider a polynomial which vanishes on the first n lines.*).
(iii) Deduce that $P(K)$ is not a proposition in the first order logic of algebraically closed fields of characteristic zero.
2. Let X be a scheme over a field k and let $x \in X$ be a point of X, with residue field k. Let

$$z = \operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

and let V be the set of all morphisms from z to X which send the unique point of z to x.
(i) Show that V is naturally a k-vector space.
(ii) Show that if $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal, then there is a natural isomorphism of k-vector spaces,

$$V \simeq \left(\frac{\mathfrak{m}}{\mathfrak{m}^2} \right)^*.$$