

HWK #4, DUE WEDNESDAY 03/02

We work over an algebraically closed field K of characteristic zero.

1. Let $p \in \mathbb{P}^3$ and let $H \subset \mathbb{P}^3$ be a plane. Show that the locus

$$\Sigma_{p,H} = \{ [l] \in \mathbb{G}(1,3) \mid p \in l \subset H \} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5,$$

is a line in \mathbb{P}^5 . Show conversely that any line lying on $\mathbb{G}(1,3)$ under the Plücker embedding is of this form.

2. Let $p \in \mathbb{P}^3$ and let $H \subset \mathbb{P}^3$ be a plane. Show that the loci

$$\Sigma_p = \{ [l] \in \mathbb{G}(1,3) \mid p \in l \} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5,$$

and

$$\Sigma_H = \{ [l] \in \mathbb{G}(1,3) \mid l \subset H \} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5,$$

are planes in \mathbb{P}^5 . Show conversely that any plane lying on $\mathbb{G}(1,3)$ under the Plücker embedding is of this form.

3. Let l_1 and $l_2 \subset \mathbb{P}^3$ be two skew lines. Let

$$Q = \{ [l] \in \mathbb{G}(1,3) \mid l \cap l_1, l \cap l_2 \neq \emptyset \} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5.$$

Show that Q is a quadric surface, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ contained in some linear subspace $\mathbb{P}^3 \subset \mathbb{P}^5$. What happens if l_1 and l_2 are not skew?

4. Now let $Q \subset \mathbb{P}^3$ be a quadric surface of rank four. Show that the two families of lines on Q correspond to two families of conics on $\mathbb{G}(1,3)$ lying on two complementary planes Λ_1 and $\Lambda_2 \subset \mathbb{P}^5$. Show that conversely the lines in \mathbb{P}^3 corresponding to a conic lying in $\mathbb{G}(1,3)$ sweep out a quadric surface provided that the plane spanned by the conic does not lie in $\mathbb{G}(1,3)$. What happens to this correspondence if either the quadric has rank three or the plane lies in $\mathbb{G}(1,3)$?

5. (a) Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be the conic given by $Z_1^2 - Z_0Z_2 = Z_3 = 0$. Find equations for the locus of lines

$$\mathcal{C}_1(C) = \{ [l] \in \mathbb{G}(1,3) \mid l \cap C \neq \emptyset \} \subset \mathbb{G}(1,3)$$

which meet C .

- (b) Same question for the twisted cubic given as the image of $[S : T] \longrightarrow [S^3 : S^2T : ST^2 : T^3]$.