HWK #3, DUE WEDNESDAY 02/23

We consider what happens if we replace $\mathbb{G}_m^n$ by $\mathbb{G}_a^n$.

Definition 0.1. *Let $\mathbb{G}$ be a connected linear algebraic group. Let X be a normal quasi-projective variety with an action of $\mathbb{G}$. We say that X is a $\mathbb{G}$-**variety**, if the stabiliser of the generic point is trivial and there is a dense orbit.*

Note that a $\mathbb{G}_m^n$-variety is the same as a toric variety.

Definition 0.2. *A **morphism** of $\mathbb{G}$-varieties is an equivariant morphism. An **isomorphism** of $\mathbb{G}$ is an isomorphism in the category of $\mathbb{G}$-varieties. A $\mathbb{G}$-**equivalence** is a commutative diagram*

$$\begin{array}{ccc} \mathbb{G} \times X_1 & \xrightarrow{(\alpha, j)} & \mathbb{G} \times X_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\ j\ } & X_2, \end{array}$$

where α is is an automorphism of $\mathbb{G}$ and j is an isomorphism.

1. Let X be a $\mathbb{G}$-variety and let $V \subset X$ be a closed subvariety, which is $\mathbb{G}$-invariant (that is, a union of $\mathbb{G}$-orbits). Show that the normalisation Y of the blow up of X along V is a $\mathbb{G}$-variety (you may assume the following; that there is an embedding of X into $\mathbb{P}^n$ and action of $\mathbb{G}$ on $\mathbb{P}^n$ which extends the action of $\mathbb{G}$ on X).
2. Show that $\mathbb{P}^n$ has the structure of a $\mathbb{G}_a^n$-variety, such that the union of the orbits which contain only one point is a hyperplane.
3. Show that if the groundfield is uncountable, then there are uncountably many inequivalent $\mathbb{G}_a^2$-varieties.
4. Show that the action

$$(a, b, [X : Y : Z]) \longrightarrow [X + aY + (b + a^2/2)Z : Y + aZ : Z],$$

gives $\mathbb{P}^2$ the structure of a $\mathbb{G}_a^2$-variety. Show that $\mathbb{P}^2$ has (at least) two inequivalent $\mathbb{G}_a^2$-structures.
5. Show that $\mathbb{P}^1 \times \mathbb{P}^1$ may be given the structure of a $\mathbb{G}_a^2$-variety. Show how to get from this $\mathbb{G}_a^2$-structure to one on $\mathbb{P}^2$ by only blowing up and blowing down unions of orbits.
Challenge problem 6. Consider the two inequivalent $\mathbb{G}_a^2$-structures on $\mathbb{P}^2$ you found in 4 (it turns out that there are only two, so this is in fact unambiguous). Show that one cannot get from from one

$\mathbb{G}_a^2$-structure to the other by blowing up and blowing down orbits of $\mathbb{G}_a^2$.