HWK #2, DUE WEDNESDAY 02/16

For questions 1 and 3 you will (probably) need to use the fact that if H is a torus acting on a quasi-projective variety with a dense orbit, then there are only finitely many orbits. You may use this fact without proving it (in fact it follows from the correspondence between cones in the fan and orbits).

1. Let $X \subset \mathbb{P}^n_K$ be an irreducible quasi-projective variety.
Let $G \subset \mathbb{P}^n_K$ be the torus, the complement of the coordinate hyperplanes. Suppose that there is a subtorus $H \subset G$ such that X is a union of finitely many orbits of H.
Prove that there are binomials $F_1, F_2, \dots, F_p$ and monomials $G_1, G_2, \dots, G_q$, such that $x \in X$ if and only if $F_i(x) = 0$ for every $1 \leq i \leq p$ and $G_j(x) \neq 0$ for some $1 \leq j \leq q$. Show that X is a toric variety if X is normal.
Finally, show that the following are equivalent:

- the natural embedding $X \subset \mathbb{P}^n_K$ is toric.
- X contains the point $[1 : 1 : 1 : \dots : 1]$ (that is, the identity of the torus)
- The equations $F_i(x) = 0$ can be put in the form monomial equals monomial.

2. Suppose that σ is a cone in $N_{\mathbb{R}} = \mathbb{R}^n$ and that $u_1, u_2, \dots, u_m$ are generators of the semigroup $S_\sigma \subset M$. Show that the affine toric variety $U_\sigma \subset \mathbb{A}^m_K$ is defined by monomial equations of the form

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} = x_1^{b_1} x_2^{b_2} \dots x_n^{b_n},$$

where

$$\sum a_i u_i = \sum b_i u_i,$$

in S_σ.

3. Prove the converse to (1). Let $X \subset \mathbb{P}^n_K$ be a normal irreducible quasi-projective variety. Suppose that there are binomials $F_1, F_2, \dots, F_p$ and monomials $G_1, G_2, \dots, G_q$, such that $x \in X$ if and only if $F_i(x) = 0$ for every $1 \leq i \leq p$ and $G_j(x) \neq 0$ for some $1 \leq j \leq q$. Prove that X is a quasi-projective toric variety.