THIRD PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature: ______

Recitation Time: ______

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) For what values of λ, μ and ν does the function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$f(x, y, z) = \lambda x^2 + \mu xy + y^2 + \nu z^2,$$

have a non-degenerate local minimum at $(0, 0, 0)$?

$$Df = (2\lambda x + \mu y,\ 2y + \mu x,\ 2\nu z)$$

$$Hf = \begin{pmatrix} 2\lambda & \mu & 0 \\ \mu & 2 & 0 \\ 0 & 0 & 2\nu \end{pmatrix}$$

$d_1 = 2\lambda$, $d_2 = 4\lambda - \mu^2$, $d_3 = 2\nu \cdot d_2$.

Minimum : $d_1 > 0$, $d_2 > 0$, $d_3 > 0$.

So $\lambda > 0$ $\mu^2 < 4\lambda$, $\nu > 0$

2. (20pts) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = 2x + y - z$

(i) Show that f has a global minimum on the ellipsoid $x^2+2y^2+3z^2 = 6$.

$K = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+2y^2+3z^2 = 6\}$

is closed + bounded. So K is compact.

f is cts, K is compact $\Longrightarrow$ f has a global minimum.

(ii) Find this minimum.

Consider $h\colon \mathbb{R}^4 \longrightarrow \mathbb{R}$ given by

$h(x,y,z,\lambda) = 2x+y-z-\lambda(x^2+2y^2+3z^2-6)$

Critical pts of h: $x = \frac{1}{\lambda}$ $\qquad x = 4y = -6z$

$4y = \frac{1}{\lambda}$ $\qquad y = \frac{-3z}{2}, \; x = -6z.$

$-6z = \frac{1}{\lambda}$

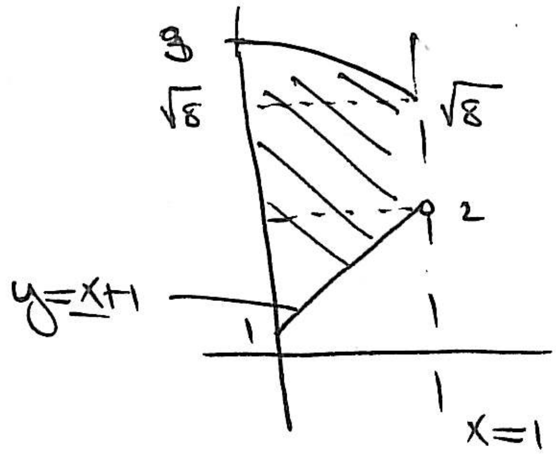
$x^2+2y^2+3z^2 = 6 \qquad z^2\left(3 + \frac{9}{2} + 36\right) = 6$

$z^2(2+3+24) = 4 \qquad z = \frac{\pm 2}{\sqrt{29}} \qquad f(P) = (24+3+2)\frac{-1}{\sqrt{29}}$

min value: + square root $\qquad P = \left(\frac{-12}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right) \qquad = -\sqrt{29}.$

3. (20pts)
(i) Draw a picture of the region of integration of

$$\int_0^1 \int_{1+x}^{\sqrt{9-x^2}} dy\, dx.$$



(ii) Change the order of integration of the integral.

$$\int_1^2 \int_0^{y-1} dx\, dy + \int_2^{\sqrt{8}} \int_0^1 dx\, dy + \int_{\sqrt{8}}^3 \int_0^{\sqrt{9-y^2}} dx\, dy$$

4. (20pts) Let W be the region inside the two cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.
Set up an integral to calculate the volume of W and calculate this integral.

View as a region of type 2.

$$\mathrm{vol}(W) = \iiint_W dx\,dy\,dz$$

$$= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \right) dz \right) dy$$

$$= 2\int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dz \right) dy$$

$$= 4\int_{-1}^{1} 1-y^2\, dy$$

$$= 4 \left[y - \frac{y^3}{3} \right]_{-1}^{1}$$

$$= 8\left(1-\frac{1}{3}\right) = \frac{16}{3}.$$

5. (20pts) Let D be the region in the first quadrant bounded by the curves $y^2 = x$, $y^2 = 2x$, $xy = 1$ and $xy = 4$.

(i) Find $du\,dv$ in terms of $dx\,dy$, where $u = \frac{y^2}{x}$ and $v = xy$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{vmatrix} = -\frac{y^2}{x} - \frac{2y^2}{x} = -\frac{3y^2}{x}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{-x}{3y^2} \qquad dx\,dy = (3u)^{-1}du\,dv.$$

(ii) Set up an integral to calculate the area of the region D and calculate this integral.

$$\iint_D dx\,dy = \int_1^4 \left(\int_1^2 \frac{du}{3u}\right) dv$$

$$= \frac{1}{3}\int_1^4 [\ln u]_1^2\, dv$$

$$= \frac{\ln 2}{3}\int_1^4 dv$$

$$= \ln 2.$$