SECOND PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Find a recursive formula for a sequence of points (x_0, y_0), (x_1, y_1), $\ldots$, (x_n, y_n), whose limit (x_∞, y_∞), if it exists, is a point of intersection of the curves

$$x^2 - y^2 = 1$$
$$x^2(x+1) = y^2.$$

Let $f(x,y)$ be the function $\mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = (x^2+y^2-1,\ x^3+x^2-y^2)$$

Then $Df(x,y) = \begin{pmatrix} 2x & -2y \\ 3x^2+2x & -2y \end{pmatrix}$ det $Df(x,y) = 6x^2y$

$$Df(x,y)^{-1} = \frac{1}{6x^2y}\begin{pmatrix} -2y & 2y \\ -3x^2-2x & 2x \end{pmatrix} \quad Df(x,y)^{-1}\begin{pmatrix} x^2-y^2-1 \\ x^2(x+1)-y^2 \end{pmatrix} =$$

$$\frac{1}{6x^2y}\begin{pmatrix} -2y & 2y \\ -3x^2-2x & 2x \end{pmatrix}\begin{pmatrix} x^2-y^2-1 \\ x^3+x^2-y^2 \end{pmatrix} = \begin{pmatrix} \frac{1+x^3}{3x^2} \\ \frac{3x^2y^2-x^4+3x^2}{6xy} \end{pmatrix}$$

So $$(x_n, y_n) = (x_{n-1}, y_{n-1}) - \left(\frac{1}{3x_{n-1}^2} + \frac{x_{n-1}}{3},\ \frac{3x_{n-1}^2y_{n-1}^2 - x_{n-1}^4 + 3x_{n-1}^2}{6x_{n-1}y_{n-1}}\right)$$

2. (20pts) Suppose that $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is differentiable at $P = (3, -2, 1)$ with derivative

$$DF(3,-2,1) = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & -3 \end{pmatrix}.$$

Suppose that $F(3, -2, 1) = (1, -3)$. Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x, y, z) = \|F(x, y, z)\|$.

(i) Show that the function $f(x, y, z)$ is differentiable at P.

Let $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(u,v) = \|(u,v)\| = (u^2+v^2)^{1/2}$

Then g is differentiable and f is the composition of F and g, $f = g \circ F$.

So f is differentiable at P.

(ii) Find $Df(3, -2, 1)$. $Df = Dg \cdot DF$

$$Dg = \frac{1}{(u^2+v^2)^{1/2}}(u,v)$$

$$Dg(1,-3) = \frac{1}{\sqrt{10}}(1,-3).$$

$$Df(3,-2,1) = Dg(1,-3) \cdot DF(3,-2,1)$$

$$= \frac{1}{\sqrt{10}}(1,-3)\begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & -3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{10}}(-5 \quad 1 \quad 12).$$

(iii) Find the directional derivative of f at P in the direction of $\hat{u} = -\frac{1}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$.

$$D_{\hat{u}} f(P) = \nabla f(P) \cdot \hat{u}$$

$$= \frac{1}{3\sqrt{10}}(-5,1,12)(-1,2,-2)$$

$$= \frac{1}{3\sqrt{10}}(5+2-24) = \frac{-17}{3\sqrt{10}}$$

3. (20pts) Let $F\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be a C^1 function. Suppose that

$$DF(3,1,0,-1) = \begin{pmatrix} 1 & 3 & 1 & 3 \\ -1 & 2 & -1 & -2 \end{pmatrix}.$$

(a) Show that there is an open subset $U \subset \mathbb{R}^2$ containing $(3,1)$ and an open subset $V \subset \mathbb{R}^2$ containing $(0,-1)$ such that for all $(x,y) \in U$, the system of equations

$$F(x,y,z,w) = F(3,1,0,-1),$$

has the unique solution

$$(z,w) = (f_1(x,y), f_2(x,y)) \qquad \text{with} \qquad (z,w) \in V.$$

The submatrix formed from the last two columns is $\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$. This has det. $-2+3 = 1 \neq 0$.

So f exists by the Implicit function Theorem

(b) Find the derivative $Df(3,1)$.

Let $G: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $G(x,y) = F(x,y,f(x,y)) - F(3,1,0,-1)$. G identically zero.

$DG = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: $Df(3,1) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

Method I $\quad 0 = \dfrac{\partial G_1}{\partial x} = \dfrac{\partial F_1}{\partial x} \cdot \dfrac{\partial x}{\partial x} + \dfrac{\partial F_1}{\partial y} \cdot \dfrac{\partial y}{\partial x} + \dfrac{\partial F_1}{\partial z} \cdot \dfrac{\partial f_1}{\partial x} + \dfrac{\partial F_1}{\partial w} \cdot \dfrac{\partial f_2}{\partial x}$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial x} \frac{\partial f_1}{\partial x} + \frac{\partial F_1}{\partial w} \cdot \frac{\partial f_2}{\partial x}$$

Plug in $(3,1)$ $\quad 0 = 1 + \alpha + 3\gamma$ $\qquad \gamma = 0, \ \alpha = -1$

$\dfrac{\partial G_2}{\partial x}(3,1)$: $\quad 0 = -1 - \alpha - 2\gamma$

Add $\quad 0 = 0 + \gamma$

Similarly $\dfrac{\partial G_1}{\partial y}(3,1)$ $\quad 0 = 3 + \beta + 3\delta$ $\qquad \delta = -5, \ \beta = 12$

$\dfrac{\partial G_2}{\partial y}(3,1)$ $\quad 0 = 2 - \beta - 2\delta$

$\quad 0 = 5 + \delta$

$$Df(3,1) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} -1 & 12 \\ 0 & -5 \end{pmatrix}$$

<u>Method II</u> Let $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ be $g(x,y) = (x, y, f(x,y))$

$G = F \circ g + (\text{constant})$ $\qquad DG = DF \cdot Dg$

$$DG(3,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ -1 & 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Rewrite as
$$\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = -\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{-2+3} \begin{pmatrix} -2 & -3 \\ +1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 12 \\ 0 & -5 \end{pmatrix} = Df(3,1)$$

4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{4}{9}\hat{\imath} - \frac{7}{9}\hat{\imath} - \frac{4}{9}\hat{k}, \quad \vec{B}(a) = \frac{1}{9}\hat{\imath} - \frac{4}{9}\hat{\imath} + \frac{8}{9}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = -\hat{\imath} + 2\hat{\jmath}.$$

Find:

(i) the unit normal vector $\vec{N}(a)$.

$$\vec{N}(a) = \vec{B}(a) \times \vec{T}(a)$$

$$= \frac{1}{9^2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -4 & 8 \\ 4 & -7 & -4 \end{vmatrix} = \frac{1}{9^2}\left[(16+56)\hat{\imath} - (-4-32)\hat{\jmath} + (-7+16)\hat{k} \right]$$

$$= \frac{8\hat{\imath}}{9} + \frac{4\hat{\jmath}}{9} + \frac{1}{9}\hat{k}$$

(ii) the curvature $\kappa(a)$.

Frenet formula: $\frac{d\vec{N}}{ds}(a) = -\kappa(a)\vec{T}(a) + \tau(a)\vec{N}(a)$

$$\kappa(a) = -\frac{d\vec{N}}{ds}(a)\cdot\vec{T}(a) = \frac{-1}{9}(-1, \mp 2, 0)\cdot(4, -7, -4) = +2$$

(iii) the torsion $\tau(a)$.

$$\tau(a) = \frac{d\vec{N}}{ds}(a)\cdot\vec{B}(a) = \frac{1}{9}(-1, -2, 0)(1, -4, 8)$$

$$= -1.$$

5. (20pts) Let $\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the vector field given by $\vec{F}(x, y) = y\hat{\imath} + x\hat{\jmath}$.

(i) Is $\vec{F}$ a gradient field (that is, is $\vec{F}$ conservative)? Why?

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = xy$.
Then $\nabla f(x,y) = y\hat{i} + x\hat{j} = \vec{F}(x,y)$ so yes $\vec{F}$ is conservative.

(ii) Is $\vec{F}$ incompressible? $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 0 = 0$

Yes, F is incompressible

(iii) Find a flow line that passes through the point $(1, 0)$.

Solve $x'(t) = y(t)$ $x(0)=1$ Let $u = x+y$, $v = x-y$ $u(t) = \alpha e^{t}$
$y'(t) = x(t)$ $y(0)=0$ $u' = u$, $v' = -v$, $u(0)=1$ $v(t) = \beta e^{-t}$
$v(0)=1$ $\alpha = \beta = 1$

$x(t) = \cosh(t) = \frac{e^{t}+e^{-t}}{2}$, $y(t) = \sinh(t) = \frac{e^{t}-e^{-t}}{2}$

(iv) Find a flow line that passes through the point (a, b), where $a^2 > b^2$.

Let $c^2 = a^2 - b^2$, $c > 0$.

Try $x(t) = c\cosh t$, $y(t) = c \sinh(t)$.
$x'(t) = y(t)$ $y'(t) = x(t)$ ✓

and ~~x(0)~~ $\vec{r}(t) = c(\cosh(t), \sinh(t))$ passes through (a,b).