FIRST PRACTICE MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS.

Signature: ____________________

Recitation Time: ____________________

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) (i) Let $\vec{u}$ and $\vec{v}$ be two vectors. Show that the vectors $\vec{a} = \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}$ and $\vec{b} = \|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}$ are orthogonal.

We check that $\vec{a} \cdot \vec{b} = 0$.

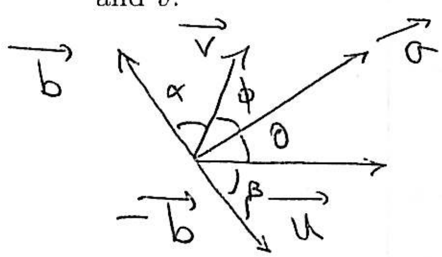
$$\vec{a} \cdot \vec{b} = \left(\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}\right)\left(\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}\right)$$

$$= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2$$

$$= 0.$$

So $\vec{a}$ and $\vec{b}$ are orthogonal.

(ii) Show that the vector $\vec{a} = \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}$ bisects the angle between $\vec{u}$ and $\vec{v}$.



We want to check $\theta = \phi$.
It suffices to check $\alpha = \beta$
as $\theta = 90 - \alpha$ and $\phi = 90 - \beta$.

Now $\vec{v} \cdot \vec{b} = \|\vec{v}\|\|\vec{b}\| \cos\alpha$

$\vec{u} \cdot -\vec{b} = \|\vec{u}\|\|\vec{b}\| \cos\beta$.

So it suffices to check $\|\vec{u}\|\, \vec{v} \cdot \vec{b} = \|\vec{v}\|\, \vec{u} \cdot -\vec{b}$.

But $\|\vec{u}\|\, \vec{v} \cdot \vec{b} = \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{v}\|\, \vec{u} \cdot \vec{v}$

and $\|\vec{v}\|\, \vec{u} \cdot -\vec{b} = \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{v}\|\, \vec{u} \cdot \vec{v}$ ✓.

2. (20pts) (i) Find the equation of the plane through the three points $P_0 = (1,1,2)$, $P_1 = (-1,2,-2)$ and $P_2 = (2,-1,1)$.

$$\overrightarrow{P_0P_1} = (-2,1,-4), \quad \overrightarrow{P_2P_0} = (1,-2,-1)$$

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_2P_0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -4 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -4 \\ -2 & -1 \end{vmatrix}\hat{i} - \begin{vmatrix} -2 & -4 \\ 1 & -1 \end{vmatrix}\hat{j} + \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\hat{k}$$

$$= -9\hat{i} - 6\hat{j} + 3\hat{k}$$

So $3\hat{i} + 2\hat{j} - \hat{k}$ normal to plane

$3(x-1) + 2(y-1) - (z-2) = 0$ is the equation of the plane.

(ii) Find the distance between this plane and the point $Q = (1,1,1)$.

Let $\underset{(x,y,z)}{\overset{\shortparallel}{R}}$ be the closest pt. Then $\overrightarrow{RQ}$ parallel to $(3,2,-1)$.

<u>Method</u> I R lies on plane and line through Q parallel to $(3,2,-1)$

$\overrightarrow{RQ} = t(3,2,-1)$ $\quad (x-1, y-1, z-1) = (3t, 2t, -t)$

$(x,y,z) = (3t+1, 2t+1, 1-t)$.

R on plane $\quad 3(3t) + 2(2t) - (1-t-2) = 0$

$14t = -1 \quad t = -\frac{1}{14} \quad R = \cancel{\frac{1}{14}(11, 12, 15)}$

$RQ = \frac{1}{14}(3,2,-1)$

distance $= \frac{1}{14}(\sqrt{14}) = \frac{1}{\sqrt{14}}$.

Method II

$$\overrightarrow{RQ} = \text{proj}_{\vec{n}} \overrightarrow{P_0Q}$$

$$= \left(\frac{\overrightarrow{P_0Q} \cdot \vec{n}}{\|\vec{n}\|^2} \right) \vec{n}$$

$$= \frac{1}{14}(3,2,-1.$$

$\overrightarrow{P_0Q} = (0,0,-1)$

$\overrightarrow{P_0Q} \cdot \vec{n} = +1.$

$\|\vec{n}\|^2 = 14$

3. (20pts) (i) What is the angle between the diagonal of a cube and one of the edges it meets?

Let the vertices of the cube by $(0,0,0)$ $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$.

Suppose the diagonal is from $(0,0,0)$ to $(1,1,1)$

Associated vector $(1,1,1)$. Meets edg $(0,0,0)$, $(1,0,0)$.

If θ angle between $(1,0,0)$ and $(1,1,1)$

$$\cos\theta = \frac{(1,0,0)\cdot(1,1,1)}{1\cdot\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \theta = \frac{\pi}{6}.$$

(ii) Find the angle between the diagonal of a cube and the diagonal of one of its faces.

Take diagonal from $(0,0,0)$ to $(1,1,0)$.

So we want angle between $(1,1,1)$ and $(1,1,0)$.

$$\cos\theta = \frac{(1,1,1)\cdot(1,1,0)}{\sqrt{2}\cdot\sqrt{3}} = \frac{2}{\sqrt{2}\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right).$$

4. (20pts) Let D be the region inside the paraboloid $a^2 z = x^2 + y^2$ and outside the sphere of radius a centred at the origin.
(i) Describe the region D in cylindrical coordinates.

Inside the paraboloid : $a^2 z \geq r^2$, $r^2 \leq a^2 z$
outside the sphere : $x^2 + y^2 + z^2 \geq a^2$
$r^2 + z^2 \geq a^2$

So $r^2 \leq a^2 z$, $r^2 + z^2 \leq a$

(ii) Describe the region D in spherical coordinates.

Inside the paraboloid: $(\rho \cos\phi)^2 \leq a^2 \rho \sin\phi$
$\rho \cos^2\phi \leq a^2 \sin\phi$

outside the sphere : $\rho \geq a$.

So $\rho \geq a$, $\rho \cos\phi \leq a^2 \tan\phi$.

5. (20pts) Determine whether or not the following limits exist, and if they do exist, then find the limit. Explain your answer.

(i) $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

Yes, the limit does exist.

$$\frac{x^4-y^4}{x^2+y^2} = x^2-y^2 \quad (x,y) \neq (0,0)$$

$$\text{So } \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x^2-y^2 = 0.$$

(ii) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$

No, the limit does not exist.

If we approach along line $x=0$

$$\lim_{y\to 0} \frac{0}{y^2} = 0.$$

If we approach along line $y=x$

$$\lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \neq 0.$$