MODEL ANSWERS TO HWK #9

1. There are a number of ways to proceed; probably the most straightforward is to view the region D as something of type 2:

$$\begin{aligned}
\iint_D x + y \, \mathrm{d}x \, \mathrm{d}y &= \int_{-1}^{2} \left(\int_{y^2-2y}^{2-y} x + y \, \mathrm{d}x \right) \mathrm{d}y \\
&= \int_{-1}^{2} \left[\frac{x^2}{2} + yx \right]_{y^2-2y}^{2-y} \mathrm{d}y \\
&= \int_{-1}^{2} \frac{(2-y)^2}{2} + y(2-y) - \frac{(y^2-2y)^2}{2} - y(y^2-2y) \, \mathrm{d}y \\
&= \int_{-1}^{2} -\frac{y^4}{2} + y^3 - \frac{y^2}{2} + 2 \, \mathrm{d}y \\
&= \left[-\frac{y^5}{2 \cdot 5} + \frac{y^4}{4} - \frac{y^3}{2 \cdot 3} + 2y \right]_{-1}^{2} \\
&= -\frac{2^4}{5} + 2^2 - \frac{2^2}{3} + 2^2 - \frac{1}{10} - \frac{1}{4} - \frac{1}{6} + 2 \\
&= \frac{99}{20}.
\end{aligned}$$

2. There are a number of ways to proceed; probably the most straightforward is to view the region D as something of type 1:

$$\begin{aligned}
\iint_D 3y \, \mathrm{d}x \, \mathrm{d}y &= \int_0^{\frac{1}{9}} \left(\int_x^3 3y \, \mathrm{d}y \right) \mathrm{d}x + \int_{\frac{1}{9}}^{1} \left(\int_x^{x^{-1/2}} 3y \, \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^{\frac{1}{9}} \left[\frac{3y^2}{2} \right]_x^3 \mathrm{d}x + \int_{\frac{1}{9}}^{1} \left[\frac{3y^2}{2} \right]_x^{x^{-1/2}} \mathrm{d}x \\
&= \int_0^{\frac{1}{9}} \frac{3^3}{2} - \frac{3x^2}{2} \, \mathrm{d}x + \int_{\frac{1}{9}}^{1} \frac{3}{2x} - \frac{3x^2}{2} \, \mathrm{d}x \\
&= \left[\frac{3^3 x}{2} - \frac{x^3}{2} \right]_0^{\frac{1}{9}} + \left[\frac{3}{2} \ln x - \frac{x^3}{2} \right]_{\frac{1}{9}}^{1} \\
&= \frac{3}{2} - \frac{1}{2 \cdot 3^6} - \frac{1}{2} + 3 \ln 3 + \frac{1}{2 \cdot 3^6} \\
&= 1 + 3 \ln 3.
\end{aligned}$$

3.

$$\begin{aligned}
\int_0^2 \left(\int_0^{4-y^2} x \, \mathrm{d}x \right) \mathrm{d}y &= \int_0^2 \left[\frac{x^2}{2} \right]_0^{4-y^2} \, \mathrm{d}y \\
&= \int_0^2 \frac{(4-y^2)^2}{2} \, \mathrm{d}y \\
&= \int_0^2 8 - 4y^2 + \frac{y^4}{2} \, \mathrm{d}y \\
&= \left[8y - \frac{4y^3}{3} + \frac{y^5}{2 \cdot 5} \right]_0^2 \\
&= 16 - \frac{32}{3} + \frac{2^4}{5} \\
&= \frac{2^4 \cdot 3 \cdot 5 - 2^5 \cdot 5 + 2^4 \cdot 3}{3 \cdot 5} \\
&= \frac{2^4 \cdot 3 \cdot 6 - 2^5 \cdot 5}{3 \cdot 5} \\
&= \frac{2^5(9-5)}{3 \cdot 5} \\
&= \frac{2^7}{3 \cdot 5}.
\end{aligned}$$

The region in question is bounded by the curves $x = 0$, $y = 0$ and $y^2 = 4 - x$. So, reversing the order of integration, we get

$$\begin{aligned}
\int_0^4 \left(\int_0^{\sqrt{4-x}} x \, \mathrm{d}y \right) \mathrm{d}x &= \int_0^4 x \, [y]_0^{\sqrt{4-x}} \, \mathrm{d}x \\
&= \int_0^4 x\sqrt{4-x} \, \mathrm{d}x \\
&= \left[-\frac{2x}{3}(4-x)^{3/2} \right]_0^4 + \int_0^4 \frac{2}{3}(4-x)^{3/2} \, \mathrm{d}x \\
&= \left[-\frac{4}{3 \cdot 5}(4-x)^{5/2} \right]_0^4 \\
&= \frac{2^7}{3 \cdot 5}.
\end{aligned}$$

4.

$$\int_0^8 \left(\int_0^{\sqrt{y/3}} y \,\mathrm{d}x \right) \mathrm{d}y + \int_8^{12} \left(\int_{\sqrt{y-8}}^{\sqrt{y/3}} y \,\mathrm{d}x \right) \mathrm{d}y = \int_0^2 \left(\int_{3x^2}^{x^2+8} y \,\mathrm{d}y \right) \mathrm{d}x$$

$$= \int_0^2 \left[\frac{y^2}{2} \right]_{3x^2}^{x^2+8} \mathrm{d}x$$

$$= \int_0^2 \frac{(x^2+8)^2}{2} - \frac{(3x^2)^2}{2} \,\mathrm{d}x$$

$$= \left[\frac{x^5}{2\cdot 5} + \frac{8x^3}{3} + 2^5 x - \frac{9x^5}{2\cdot 5} \right]_0^2$$

$$= \frac{2^4}{5} + \frac{2^6}{3} + 2^6 - \frac{9\cdot 2^4}{5}$$

$$= \frac{896}{15}.$$

5. This is a region of type 4; we view this as an elementary region of type 1. The projection of W onto the xy-plane is the elementary region of type 2 bounded by $y = x^2$ and $y = 9$.

$$\iiint_W 8xyz \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{-3}^{3} \left(\int_{x^2}^{9} \left(\int_0^{9-y} 8xyz \,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x$$

$$= 8 \int_{-3}^{3} x \left(\int_{x^2}^{9} y \left(\int_0^{9-y} z \,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x$$

$$= 8 \int_{-3}^{3} x \left(\int_{x^2}^{9} y \left[\frac{z^2}{2} \right]_0^{9-y} \mathrm{d}y \right) \mathrm{d}x$$

$$= 8 \int_{-3}^{3} x \left(\int_{x^2}^{9} \frac{y(9-y)^2}{2} \,\mathrm{d}y \right) \mathrm{d}x$$

$$= 4 \int_{-3}^{3} x \left(\int_{x^2}^{9} 81y - 18y^2 + y^3 \,\mathrm{d}y \right) \mathrm{d}x$$

$$= 4 \int_{-3}^{3} x \left[\frac{81y^2}{2} - 6y^3 + \frac{y^4}{4} \right]_{x^2}^{9} \mathrm{d}x$$

$$= 4 \int_{-3}^{3} \left(\frac{3^8}{2} - 2\cdot 3^7 + \frac{3^8}{4} \right) x - \frac{81x^3}{2} + 6x^7 - \frac{x^9}{4} \mathrm{d}x$$

$$= 0,$$

as x, x^3, x^7 and x^9 are all odd functions. In retrospect, we could have decide very early on that the integral is zero;

$$J(x) = \int_{x^2}^{9} y \left(\int_0^{9-y} z \, \mathrm{d}z \right) \mathrm{d}y,$$

is clearly an even function of x, so that $xJ(x)$ is an odd function.

6. This is a region of type 4; we view this as an elementary region of type 1. The projection of W onto the xy-plane is the elementary region of type 2 bounded by $x = 0$, $y = 3$ and $y = x$.

$$\begin{aligned}
\iiint_W z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \int_0^3 \left(\int_x^3 \left(\int_0^{\sqrt{9-y^2}} z \, \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^3 \left(\int_x^3 \left[\frac{z^2}{2} \right]_0^{\sqrt{9-y^2}} \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^3 \left(\int_x^3 \frac{9 - y^2}{2} \, \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{2} \int_0^3 \left[9y - \frac{y^3}{3} \right]_x^3 \mathrm{d}x \\
&= \frac{1}{2} \int_0^3 18 - 9x + \frac{x^3}{3} \, \mathrm{d}x \\
&= \frac{1}{2} \left[18x - \frac{9x^2}{2} + \frac{x^4}{12} \right]_0^3 \\
&= 3^3 - \frac{3^4}{4} + \frac{3^3}{8} \\
&= \frac{3^3}{8}(8 - 6 + 1) \\
&= \frac{81}{8}.
\end{aligned}$$

7. This is the region bounded by the planes $y = \pm 1$, $x = y^2$, $z = 0$ and $x + z = 1$. So the other five ways to write this region are:

$$
\begin{aligned}
&\int_0^1 \left(\int_{-\sqrt{x}}^{\sqrt{x}} \left(\int_0^{1-x} f(x,y,z)\, \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&\int_0^1 \left(\int_0^{1-x} \left(\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\, \mathrm{d}y \right) \mathrm{d}z \right) \mathrm{d}x \\
&\int_0^1 \left(\int_0^{1-z} \left(\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\, \mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z \\
&\int_{-1}^1 \left(\int_0^{1-y^2} \left(\int_{y^2}^{1-z} f(x,y,z)\, \mathrm{d}x \right) \mathrm{d}z \right) \mathrm{d}y \\
&\int_0^1 \left(\int_{\sqrt{1-z}}^{\sqrt{1-z}} \left(\int_{y^2}^{1-z} f(x,y,z)\, \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z.
\end{aligned}
$$

8. T is a linear transformation; therefore it takes straight lines to straight lines. So D is the parallelogram with vertices

$$T(0,0) = (0,0) \quad T(1,3) = (11,2) \quad T(-1,2) = (4,3) \quad T(0,5) = (15,5).$$

9. Since T is supposed to take $(0,5)$ to $(4,1)$, it must take $(0,1)$ to $(4/5, 1/5)$. Since T is supposed to take $(-1,3)$ to $(3,2)$ and $(1,2)$ to $(1,-1)$ it should take

$$(5,0) = 3(1,2) - 2(-1,3),$$

to

$$3(3,2) - 2(1,-1) = (7,8).$$

Therefore

$$T(1,0) = (7/5, 8/5).$$

Therefore

$$T(u,v) = \begin{pmatrix} 7/5 & 4/5 \\ 8/5 & 1/5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

10. We have $x = u$ and $y = (v+u)/2$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \begin{vmatrix} 1 & 0 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore

$$\int_0^2 \left(\int_{x/2}^{(x/2)+1} x^5(2y-x)e^{(2y-x)^2}\,\mathrm{d}x \right) \mathrm{d}y = \frac{1}{2}\int_0^2 \left(\int_0^2 u^5 v e^{v^2}\,\mathrm{d}v \right) \mathrm{d}u$$
$$= \frac{1}{4}\int_0^2 u^5 \left[e^{v^2} \right]_0^2 \mathrm{d}u$$
$$= \frac{e^4-1}{4}\int_0^2 u^5 \mathrm{d}u$$
$$= \frac{e^4-1}{24}\left[u^6 \right]_0^2$$
$$= \frac{8(e^4-1)}{3}.$$

11. Let $u = 2x + y$ and $v = x - y$. Then

$$\frac{\partial(u,x)}{\partial(x,y)}(x,y) = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3.$$

So

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = -\frac{1}{3}.$$

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore,

$$\iint_D (2x+y)^2 e^{x-y}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{3}\int_1^4 \left(\int_{-1}^1 u^2 e^v\,\mathrm{d}v \right) \mathrm{d}u$$
$$= \frac{1}{3}\int_1^4 u^2 \left[e^v \right]_{-1}^1 \mathrm{d}u$$
$$= \frac{e-e^{-1}}{3}\int_1^4 u^2\,\mathrm{d}u$$
$$= \frac{e-e^{-1}}{9}\left[u^3 \right]_1^4$$
$$= 7(e-e^{-1}).$$

12. Let $u = y + 2x$ and $v = 2y - x$. Then D^* is the region

$$[0,5] \times [-5,0],$$

in uv-coordinates.

$$\frac{\partial(u,x)}{\partial(x,y)}(x,y) = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5.$$

So

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \frac{1}{5}.$$

This is nowhere zero. As the map is linear, it follows that the map is injective, and so by the Inverse function theorem it defines a diffeomorphism. Therefore

$$\begin{aligned}
\iint_D \frac{2x+y-3}{2y-x+6}\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{5}\int_0^5 \left(\int_{-5}^0 \frac{u-3}{v+6}\,\mathrm{d}v\right)\mathrm{d}u \\
&= \frac{1}{5}\int_0^5 (u-3)\left[\ln(v+6)\right]_{-5}^0\,\mathrm{d}u \\
&= \frac{\ln 6}{5}\int_0^5 (u-3)\,\mathrm{d}u \\
&= \frac{\ln 6}{5}\left[\frac{u^2}{2} - 3u\right]_0^5 \\
&= \ln 6\left(\frac{5}{2} - 3\right) \\
&= -\frac{\ln 6}{2}.
\end{aligned}$$