MODEL ANSWERS TO HWK #7
(18.022 FALL 2010)

(1) (a) F is a gradient field given by the potential $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C$.
 (b) The flow line $r(t) = (x(t), y(t))$ satisfies $x'(t) = x(t)$, $y'(t) = y(t)$ and $x(0) = a$, $y(0) = b$. The solution is $x(t) = A_1e^t$ and $y(t) = A_2e^t$ with $A_1 = a$ and $A_2 = b$.

(2) (3.3.2) See Figure 1 below.

(3) (3.3.21) The flow line $(x(t), y(t))$ satisfies $x'(t) = x^2(t)$ and $y'(t) = y$ with initial conditions $(x(1), y(1)) = (1, e)$. Solving gives that $x(t) = \frac{1}{2-t}$ and $y(t) = e^t$.

(4) (3.3.24) The potential is $f(x, y, z) = x^2 + y^2 - 3z$.

(5) (4.1.8) We compute partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2+y^2+1)^2}|(0,0) = 0\,, \quad \frac{\partial f}{\partial y} = \frac{-2y}{(x^2+y^2+1)^2}|(0,0) = 0\,,$$
$$\frac{\partial^2 f}{\partial x \partial x} = \frac{-2(x^2+y^2+1)^2 - 4x^2}{(x^2+y^2+1)^4}|(0,0) = -2\,, \frac{\partial^2 f}{\partial y \partial y}(0,0) = -2\,,$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-8xy}{(x^2+y^2+1)^3}|(0,0) = 0\,.$$

Hence the Taylor polynomial is

$$P(x,y) = 1 - x^2 - y^2\,.$$

(6) (4.1.14) By (4.1.8) the Hessian at $(0,0)$ is

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

(7) (4.1.18) The first order derivatives are

$$f_x = 3x^2 + 2xy\,, \quad f_y = x^2 - z^2\,, \quad f_z = 6z^2 - 2yz\,.$$

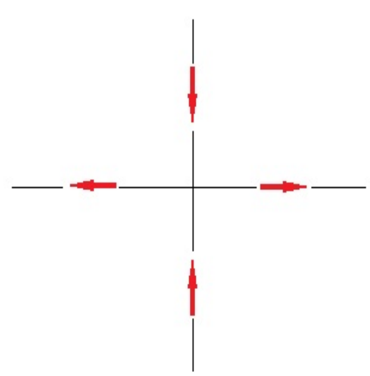

FIGURE 1. Vector field for (3.3.2)

The second order derivatives are

$$f_{xx} = 6x + 2y\,, \quad f_{yy} = 0\,, \quad f_{zz} = 12z - 2y\,, \quad f_{xy} = 2x\,, \quad f_{xz} = 0\,, \quad f_{yz} = -2z\,.$$

Hence the derivatives matrix at $(1, 0, 1)$ is

$$Df(1,0,1) = (3, 0, 6)\,,$$

and the Hessian is

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{pmatrix}$$

So

$$P_2(x,y,z) = 3 + 3(x-1) + 6(z-1) + 3(x-1)^2 + 6(z-1)^2 + 2(x-1)y - 2y(z-1)\,.$$

(8) (4.1.20) First order derivatives are

$$f_x = f\,, \quad f_y = 2f\,, \quad f_z = 3f\,.$$

Second order derivatives are

$$f_{xx} = f\,, \quad f_{xy} = 2f\,, \quad f_{xz} = 3f\,, \quad f_{yz} = 6f\,, \quad f_{yy} = 4f\,, \quad f_{zz} = 9f\,.$$

For the third order it is convenient to write (x_1, x_2, x_3) for (x, y, z) and then for any i, j, k in $\{1, 2, 3\}$ we have

$$f_{x_i x_j x_k} = ijkf\,.$$

Since $f(0, 0, 0) = 1$ we have

$$\begin{aligned} P_3(x,y,z) &= 1 + x + 2y + 3z + \frac{x^2}{2} + 2y^2 + \frac{9z^2}{2} + 2xy + 3xz + 6yz \\ &+ \frac{x^3}{6} + \frac{8y^3}{6} + \frac{27z^3}{6} + 6xyz + x^2y + \frac{9x^2z}{6} + 2y^2x + 6y^2z + \frac{9z^2x}{2} + 9z^2y\,. \end{aligned}$$

(9) (4.1.33)

(a) First order derivatives are

$$f_x = -\sin x \sin y\,, \quad f_y = \cos x \cos y\,,$$

so they both vanish at $(0, \pi/2)$. Second order derivatives are

$$f_{xx} = -\cos x \sin y\,, \quad f_{yy} = -\cos x \sin y\,, \quad f_{xy} = -\sin x \cos y\,,$$

So

$$P_2(x,y) = 1 - \frac{x^2}{2} - \frac{(y - \pi/2)^2}{2}\,.$$

(b) We use the Lagrange form of the remainder. There are 8 terms in the formula. In each term, the absolute value of the third order derivative is at most 1 (because it is some combination of cos and sin) and $h_i h_j h_k$ is at most $(0.3)^3$. Hence the remainder is at most $\frac{8}{6}(0.3)^3 = 0.036$.

(10) (4.1.34)

(a) First order derivatives are

$$f_x = f\,, \quad f_y = 2f\,,$$

so at the origin their values are $(1, 2)$. The second order derivatives are

$$f_{xx} = f\,, \quad f_{yy} = 4f\,, \quad f_{xy} = 2f\,.$$

So

$$P_2(x, y) = 1 + x + 2y + \frac{x^2}{2} + 2y^2 + 2xy\,.$$

(b) Note that $f_{yyy} = 8f$ is the largest of the third order partial derivatives. As before, we have 8 terms, and each is at most $e^{0.3}(0.1)^3$ so the remainder is at most $\frac{8}{6}8e^{0.3}(0.1)^3$. Note that you can get a better bound if you consider each third order derivative separately.