MODEL ANSWERS TO HWK #6
(18.022 FALL 2010)

(1) The curve C is given in rectangular coordinates by $\vec{r}(\theta) = (f(\theta)\cos(\theta), f(\theta)\sin(\theta))$. Then

$$\vec{r}'(\theta) = (f'(\theta)\cos(\theta) - f(\theta)\sin(\theta), f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)),$$

and the arc length of C is given by

$$\begin{aligned} s(\theta) &= \int_\alpha^\theta \|\vec{r}'(\tau)\|\, d\tau \\ &= \int_\alpha^\theta \sqrt{(f'(\tau)\cos(\tau) - f(\tau)\sin(\tau))^2 + (f'(\tau)\sin(\tau) + f(\tau)\cos(\tau))^2} d\tau \\ &= \int_\alpha^\theta \sqrt{f(\tau)^2 + f'(\tau)^2} d\tau. \end{aligned}$$

(2) (3.1.18)
At $t = 1$, the path $\mathbf{x}(t) = (\cos(e^t), 3t^2, t)$ passes in point $\mathbf{x}(1) = (\cos(e), 3, 1)$ and has velocity $\mathbf{x}(1) = (-e^t\sin(e^t), 6t, 1)|_{t=1} = (-e\sin(e), 6, 1)$. Thus, the line tangent to the path at $t = 1$ is

$$\begin{aligned} l(t) &= (\cos(e), 3, 1) + (t-1)(-e\sin(e), 6, 1) \\ &= (\cos(e) + e\sin(e) - t\sin(e), 6t - 3, t) \end{aligned}$$

(3) (3.1.26)
(a) For the balls to collide, they have to be at the same point at the same time: $t^2 - 2 = t$ and $\frac{t^2}{2} - 1 = 5 - t^2$, which solving for t yields $t = 2$, and $\mathbf{x}(2) = \mathbf{y}(2) = (2, 1)$.
(b) We have to find the angle between $\mathbf{x}'(2)$ and $\mathbf{y}'(2)$. We have $\mathbf{x}'(2) = (2t, t)|_{t=2} = (4, 2)$ and $\mathbf{y}'(2) = (1, -2t)|_{t=2} = (1, -4)$, so the angle is

$$\text{arc cos}\left(\frac{(4,2)\cdot(1,-4)}{\|(4,2)\|\,\|(1,-4)\|}\right) = \text{arc cos}\left(\frac{-4}{\sqrt{20}\sqrt{17}}\right) = \text{arc cos}\left(\frac{-2}{\sqrt{85}}\right) \approx 1.79 \text{ rad.}$$

(4) (3.1.30)
(a) We want to show that $\|\mathbf{x}(t)\| = 1$, or equivalently $\|\mathbf{x}(t)\|^2 = 1$:

$$\begin{aligned} \|\mathbf{x}(t)\|^2 &= \cos^2 t + \cos^2 t \sin^2 t + \sin^4 t \\ &= \cos^2 t + (\cos^2 t + \sin^2 t)\sin^2 t \\ &= \cos^2 t + \sin^2 t \\ &= 1. \end{aligned}$$

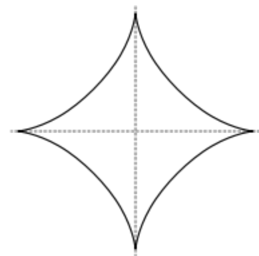

FIGURE 1. Astroid of Problem 6

(b) We want to show that $\mathbf{x}(t) \cdot \mathbf{v}(t) = 0$, where the velocity vector is $\mathbf{v}(t) = \mathbf{x}'(t) = (-\sin t, -\sin^2 t + \cos^2 t, 2\sin t \cos t)$:

$$\begin{aligned}\mathbf{x}(t) \cdot \mathbf{v}(t) &= -\cos t \sin t - \cos t \sin^3 t + \cos^3 t \sin t + 2\sin^3 t \cos t \\ &= \cos t \sin t(-1 + \sin^2 t + \cos^2 t) \\ &= 0.\end{aligned}$$

(c) If $\mathbf{x}(t)$ is a differentiable path that lies on a sphere centered at the origin, then $\mathbf{x}(t)$ has constant length equal to the radius of that sphere. Proposition 1.7 then tells us that for all values of the parameter t, the position vector $\mathbf{x}(t)$ is perpendicular to its derivative $\frac{d\mathbf{x}(t)}{dt}$, which is the velocity vector $\mathbf{v}(\mathbf{t})$.

(5) (3.1.32)

The function $\|\mathbf{x}(t)\|^2$ has a minimum at t_0, so its derivative must vanish:

$$\frac{d\,\|\mathbf{x}(t)\|^2}{dt}|_{t=t_0} = 2\mathbf{x}(t_0) \cdot \mathbf{x}'(t_0) = 0.$$

(6) (3.2.7)

For a sketch of the curve, see Figure 1.

The velocity vector and the speed for this path are

$$\begin{aligned}\mathbf{x}'(t) &= (-3a\cos^2 t \sin t, 3a \sin^2 t \cos t) \\ \|\mathbf{x}'(t)\| &= \sqrt{9a^2 \cos^4 t \sin^2 t + 9a \sin^4 t \cos^2 t} = 3a \sin t \cos t.\end{aligned}$$

Since the curve is piecewise C^1, the length of the total curve is the sum of the lengths of the four smooth pieces, or since the pieces are all congruent, the total length is

$$L = 4\int_0^{\frac{\pi}{2}} \|\mathbf{x}'(t)\|\,dt = 12a\left(\frac{\sin^2 t}{2}\right)_{t=0}^{\frac{\pi}{2}} = 6a.$$

(7) (3.2.12)

(a) The velocity vector and speed for this path are

$$\begin{aligned}\mathbf{x}'(t) &= (e^{at}(a\cos(bt) - b\sin(bt)), e^{at}(a\sin(bt) + b\cos(bt)), ae^{at}) \\ \|\mathbf{x}'(t)\| &= e^{at}\sqrt{(a\cos(bt) - b\sin(bt))^2 + (a\sin(bt) + b\cos(bt))^2 + a^2} \\ &= e^{at}\sqrt{2a^2 + b^2}.\end{aligned}$$

The arc length parameter is

$$s(t) = \int_0^t e^{a\tau}\sqrt{2a^2+b^2}d\tau = \frac{e^{at}-1}{a}\sqrt{2a^2+b^2} = (e^{at}-1)\sqrt{2+\left(\frac{b}{a}\right)^2}$$

(b) Solving $s = (e^{at}-1)\sqrt{2+\left(\frac{b}{a}\right)^2}$ for t we get $t = \frac{1}{a}\log\Delta(s)$, where $\Delta(s) = 1+\frac{s}{\sqrt{2+\left(\frac{b}{a}\right)^2}}$.

Then,

$$\mathbf{x}(s) = \Delta(s)\left(\cos(\frac{b}{a}\log\Delta(s)), \sin(\frac{b}{a}\log\Delta(s)), 1\right).$$

(8) (a)Differentiating and using the Frenet-Serret formulas $\vec{T}'(s) = \kappa(s)\vec{N}(s)$, $\vec{N}'(s) = -\kappa(s)\vec{T}(s)+\tau(s)\vec{B}(s)$ and $\vec{B}'(s) = -\tau(s)\vec{N}(s)$ we obtain

$$\frac{d}{ds}\left(\left\|\vec{T}_1(s)-\vec{T}_2(s)\right\|^2 + \left\|\vec{N}_1(s)-\vec{N}_2(s)\right\|^2 + \left\|\vec{B}_1(s)-\vec{B}_2(s)\right\|^2\right) =$$
$$= 2\left((\vec{T}_1-\vec{T}_2)\cdot(\vec{T}_1'-\vec{T}_2') + (\vec{N}_1-\vec{N}_2)\cdot(\vec{N}_1'-\vec{N}_2') + (\vec{B}_1-\vec{B}_2)\cdot(\vec{B}_1'-\vec{B}_2')\right) =$$
$$= -2\left(\kappa(\vec{T}_1\cdot\vec{N}_2+\vec{T}_2\cdot\vec{N}_1) - \kappa(\vec{N}_1\cdot\vec{T}_2+\vec{N}_2\cdot\vec{T}_1) + \tau(\vec{N}_1\cdot\vec{B}_2+\vec{N}_2\cdot\vec{B}_1) - \tau(\vec{B}_1\cdot\vec{N}_2+\vec{B}_2\cdot\vec{N}_1)\right) =$$
$$= 0.$$

Since the derivative with respect to s is zero, the quantity above is constant as a function of s.

(b) At $s = a$ the quantity above is equal to zero. But because it constant as a function of s, it must be constant equal to zero. It follows that for all s, we have $\vec{T}_1(s) = \vec{T}_2(s)$ (and also $\vec{N}_1(s) = \vec{N}_2(s)$ and $\vec{B}_1(s) = \vec{B}_2(s)$). Since we can get the position vectors of the paths $\vec{r}_i$ $(i = 1, 2)$ by integrating the velocity vector $\vec{T}_i(s)$, they must coincide:

$$\vec{r}_1(s) = \vec{r}_1(a) + \int_a^s \vec{T}_1(t)dt = \vec{r}_2(a) + \int_a^s \vec{T}_2(t)dt = \vec{r}_2(s).$$

(9) (a) We simply have to show that $\|\vec{r}'(s)\| = 1$:

$$\|\vec{r}'(s)\| = \sqrt{(-\frac{a}{c}\sin(s/c))^2 + (\frac{a}{c}\cos(s/c))^2 + (\frac{b}{c})^2} = \sqrt{\frac{a^2+b^2}{c^2}} = 1.$$

(b)

$$\vec{T}(s) = \vec{r}'(s) = (-\frac{a}{c}\sin(s/c), \frac{a}{c}\cos(s/c), \frac{b}{c})$$

$$\vec{N}(s) = \frac{d\vec{T}(s)/ds}{\left\|d\vec{T}(s)/ds\right\|} = \frac{(-\frac{a}{c^2}\cos(s/c), -\frac{a}{c^2}\sin(s/c), 0)}{\frac{a}{c^2}} = (-\cos(s/c), -\sin(s/c), 0)$$

$$\vec{B}(s) = \vec{T}(s)\times\vec{N}(s) = \begin{vmatrix} i & j & k \\ -\frac{a}{c}\sin(s/c) & \frac{a}{c}\cos(s/c) & \frac{b}{c} \\ -\cos(s/c) & -\sin(s/c) & 0 \end{vmatrix} = (\frac{b}{c}\sin(s/c), -\frac{b}{c}\cos(s/c), \frac{a}{c})$$

(c)

$$\kappa(s) = \left\| \frac{d\vec{T}(s)}{ds} \right\| = \frac{a}{c^2}$$

$$\begin{aligned}
&\frac{d\vec{B}(s)}{ds} = -\tau(s)\vec{N}(s) \\
\iff &(\frac{b}{c^2}\cos(s/c), \frac{b}{c^2}\sin(s/c), 0) = -\tau(s)(-\cos(s/c), -\sin(s/c), 0) \\
\iff &\tau(s) = \frac{b}{c^2}
\end{aligned}$$

(10) The helix in Problem 9 above has constant curvature and torsion, and by Theorem 2.5, any curve with constant curvature and torsion is congruent to such a helix. To find out which helix we solve for a, b and c the following equations:

$$\kappa = \frac{a}{c^2} \quad , \quad \tau = \frac{b}{c^2} \quad , \quad a^2 + b^2 = c^2.$$

Writing $\kappa^2 + \tau^2 = \frac{a^2}{c^4} + \frac{b^2}{c^4} = \frac{1}{c^2}$, we conclude that

$$a = \frac{\kappa}{\kappa^2 + \tau^2} \quad , \quad b = \frac{\tau}{\kappa^2 + \tau^2} \quad , \quad c = \frac{1}{\sqrt{\kappa^2 + \tau^2}}.$$

(11) (a) The vectors $\vec{T}(a), \vec{N}(a)$ and $\vec{B}(a)$ are mutually orthogonal and all have length 1, so we must have

$$\vec{N}(a) = \vec{B}(a) \times \vec{T}(a) = (\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}) \times (\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}) = (\frac{-2}{3}, \frac{1}{3}, \frac{-2}{3}).$$

(b) Dotting with $\vec{T}(s)$ on both sides of the Frenet-Serret formula

$$\frac{d\vec{N}(s)}{ds} = -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s)$$

we obtain $\frac{d\vec{N}(s)}{ds} \cdot \vec{T}(s) = -\kappa(s)$, and so $\kappa(a) = -(-4, 2, 5) \cdot (\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}) = 3$.

(c) Dotting with $\vec{B}(s)$ on both sides of the Frenet-Serret formula

$$\frac{d\vec{N}(s)}{ds} = -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s)$$

we obtain $\frac{d\vec{N}(s)}{ds} \cdot \vec{B}(s) = \tau(s)$, and so $\tau(a) = (-4, 2, 5) \cdot (\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}) = 6$.