# MODEL ANSWERS TO HWK #5

1. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function given by $f(x,y) = (x^2+y^2-1, y^2-x^2(x+1))$. Then we are looking for solutions to the equation

$$f(x,y) = (0,0).$$

We compute the derivative of $f$,

$$Df(x,y) = \begin{pmatrix} 2x & 2y \\ -2x-3x^2 & 2y \end{pmatrix}.$$

The determinant is then

$$4xy + 2xy(2+3x) = 2xy(4+3x).$$

Therefore the inverse matrix to the derivative of $f$ is

$$Df(x,y)^{-1} = \frac{1}{2xy(4+3x)} \begin{pmatrix} 2y & -2y \\ 2x+3x^2 & 2x \end{pmatrix}.$$

So we want

$$\begin{aligned} Df(x,y)^{-1}f(x,y) &= \frac{1}{2xy(4+3x)} \begin{pmatrix} 2y & -2y \\ 2x+3x^2 & 2x \end{pmatrix} \begin{pmatrix} x^2+y^2-1 \\ y^2-x^2-x^3 \end{pmatrix} \\ &= \frac{1}{2xy(4+3x)} \begin{pmatrix} 4x^2y-2y+2yx^3 \\ x^4+4xy^2+3x^2y^2-2x-3x^2 \end{pmatrix}. \end{aligned}$$

It follows that the recursion is

$$\begin{aligned} (x_1,y_1) &= (x_0,y_0) - (\frac{2x_0^2-1+x_0^3}{2x_0(4+3x_0)}, \frac{x_0^3+4y_0^2+3x_0y_0^2-2-3x_0}{4y_0(4+3x_0)}) \\ (x_2,y_2) &= (x_1,y_1) - (\frac{2x_1^2-1+x_1^3}{2x_1(4+3x_1)}, \frac{x_1^3+4y_1^2+3x_1y_1^2-2-3x_1}{4y_1(4+3x_1)}) \\ \vdots &= \vdots \\ (x_n,y_n) &= (x_{n-1},y_{n-1}) - (\frac{2x_{n-1}^2-1+x_{n-1}^3}{2x_{n-1}(4+3x_{n-1})}, \frac{x_{n-1}^3+4y_{n-1}^2+3x_{n-1}y_{n-1}^2-2-3x_{n-1}}{4y_{n-1}(4+3x_{n-1})}) \\ \vdots &= \vdots. \end{aligned}$$

2. We have

$$Df(x,y) = \nabla f = (f_x, f_y) = (-y\sin xy + 3x^2, -x\sin xy + 2y).$$

Let $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the function given by $g(x, y) = (-y \sin xy + 3x^2, -x \sin xy + 2y)$. Then

$$Dg(x, y) = \begin{pmatrix} -y^2 \cos xy + 6x & -\sin xy - xy \cos xy \\ -\sin xy - xy \cos xy & -x^2 \cos xy + 2 \end{pmatrix}.$$

The determinant of $Dg(x, y)$ is

$$\begin{aligned} d &= x^2y^2 \cos^2 xy - 6x^3 \cos xy - 2y^2 \cos xy + 12x - (\sin xy + xy \cos xy)^2 \\ &= -(6x^3 + 2y^2) \cos xy + 12x - \sin^2 xy - \sin 2xy. \end{aligned}$$

So the inverse of the derivative of $g$ is

$$Dg(x, y)^{-1} = \frac{1}{d} \begin{pmatrix} -x^2 \cos xy + 2 & \sin xy + xy \cos xy \\ \sin xy + xy \cos xy & -y^2 \cos xy + 6x \end{pmatrix}.$$

In this case

$$\begin{aligned} &Dg(x, y)^{-1} g(x, y) \\ &= \frac{1}{d} \begin{pmatrix} -x^2 \cos xy + 2 & \sin xy + xy \cos xy \\ \sin xy + xy \cos xy & -y^2 \cos xy + 6x \end{pmatrix} \begin{pmatrix} -y \sin xy + 3x^2 \\ -x \sin xy + 2y \end{pmatrix} \\ &= \frac{1}{d} \begin{pmatrix} (-x^2 \cos xy + 2)(-y \sin xxy + y^2) + (\sin xy + xy \cos xy)(-y^2 \cos xy + 6x) \\ (\sin xy + xy \cos xy)(-y \sin xy + 3x^2) + (-y^2 \cos xy + 6x)(-x \sin xy + 2y) \end{pmatrix} \\ &= (X(x, y), Y(x, y)). \end{aligned}$$

Thus the recursion is given by

$$\begin{aligned} (x_1, y_1) &= (x_0, y_0) - (X(x_0, y_0), Y(x_0, y_0)) \\ (x_2, y_2) &= (x_1, y_1) - (X(x_1, y_1), Y(x_1, y_1)) \\ \vdots &= \vdots \\ (x_n, y_n) &= (x_{n-1}, y_{n-1}) - (X(x_1, y_1), Y(x_1, y_1)) \\ \vdots &= \vdots. \end{aligned}$$

3. (a) The composite is differentiable at $(-2, 1)$ by Theorem 12.1 of the notes (or Theorem 5.3 of the book).
(b)

$$Dg(y_1, y_2) = (2y_1, -2y_2) \qquad \text{so that} \qquad Dg(1, 3) = (2, -6).$$

The chain rule says that

$$\begin{aligned} D(g \circ f)(-2, 1) &= Dg(1, 3) Df(-2, 1) \\ &= (2, -6) \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \\ &= (2, 0). \end{aligned}$$

4. (a) Note the submatrix

$$\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$$

formed by taking the last two columns of the derivative is an invertible matrix (the determinant is $1 - 4 = -3$). The result we want is then a consequence of the implicit function theorem.
(b) Let $g\colon \mathbb{R} \longrightarrow \mathbb{R}^2$ be the function $g(x) = F(x, f(x))$. Then $g(x) = (0, 0)$, so that

$$\frac{dg_1}{dx} = 0 \qquad \text{and} \qquad \frac{dg_2}{dx} = 0.$$

On the other hand, the chain rule says,

$$\begin{aligned} \frac{dg_1}{dx} &= \frac{\partial F_1}{\partial x}\frac{dx}{dx} + \frac{\partial F_1}{\partial y_1}\frac{df_1(x)}{dx} + \frac{\partial F_1}{\partial y_2}\frac{df_2(x)}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y_1}\frac{df_1(x)}{dx} + \frac{\partial F_1}{\partial y_2}\frac{df_2(x)}{dx} \end{aligned}$$

So, plugging in the point $(4, -1, 2)$, we get

$$0 = 1 - 1(\frac{df_1(x)}{dx})(4) + 4(\frac{df_2(x)}{dx})(4).$$

Similarly,

$$\begin{aligned} \frac{dg_2}{dx} &= \frac{\partial F_2}{\partial x}\frac{dx}{dx} + \frac{\partial F_2}{\partial y_1}\frac{df_1(x)}{dx} + \frac{\partial F_2}{\partial y_2}\frac{df_2(x)}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y_1}\frac{df_1(x)}{dx} + \frac{\partial F_2}{\partial y_2}\frac{df_2(x)}{dx} \end{aligned}$$

So, plugging in the point $(4, -1, 2)$, we get

$$0 = 0 + 1(\frac{df_1(x)}{dx})(4) - 1(\frac{df_2(x)}{dx})(4).$$

This gives us two linear equations in two unknowns:

$$\begin{aligned} -a + 4b &= -1 \\ a - b &= 0. \end{aligned}$$

Adding these two equations, we get

$$3b = -1,$$

so that

$$b = -1/3 \qquad \text{and} \qquad a = 1/3.$$

Hence

$$\frac{df_1(x)}{dx}(4) = 1/3 \qquad \text{and} \qquad \frac{df_2(x)}{dx}(4) = -1/3,$$

so that

$$Df(4) = (1/3, -1/3).$$

Here is a slightly more slick way of finding the derivative. The function $g$ is the composition of the function $F$ and the function

$$h\colon \mathbb{R} \longrightarrow \mathbb{R}^3,$$

given by $h(x) = (x, f(x)) = (x, f_1(x), f_2(x))$. So

$$Dg(x) = D(F \circ h) = DF(x, f(x))Dh(x).$$

Note that

$$Dh = \begin{pmatrix} 1 \\ \frac{df_1}{dx} \\ \frac{df_2}{dx} \end{pmatrix}.$$

Plugging in the point $x = 4$, we get

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = Dg(4) = DF(4, -1, 2)Dh(4) = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{df_1}{dx}(4) \\ \frac{df_2}{dx}(4) \end{pmatrix}.$$

Multiplying out we get the same pair of simultaneous linear equations and we can now continue as above.

5. (a) Let $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function given by $F(x, y, z) = x^3y^3 + y^3z^3 + z^3x^3 - 1$. Note that $F(x, y, z) = 0$ if and only if $(x, y, z) = 0$. Now

$$\begin{aligned} DF(2, -1, 1) &= 3(x^2(y^3 + z^3), y^2(x^3 + z^3), z^2(y^3 + x^3))\Big|_{(2,-1,1)} \\ &= 3(0, 9, 7). \end{aligned}$$

As the submatrix formed by taking the last column of the derivative is an invertible matrix (that is (21) is an invertible $1 \times 1$ matrix), the result we want is then a consequence of the implicit function theorem.

(b) Define a function

$$g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3,$$

by the rule $g(x, y) = F(x, y, f(x, y)$. As $g(x, y) = 0$, we have

$$\frac{\partial g}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial g}{\partial y} = 0.$$

By the chain rule,

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial x} \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial x}. \end{aligned}$$

It follows that

$$\frac{\partial f}{\partial x}(2,-1) = -\frac{\frac{\partial F}{\partial x}(2,-1,1)}{\frac{\partial F}{\partial z}(2,-1,1)} = 0.$$

Similarly

$$\frac{\partial f}{\partial y}(2,-1) = -\frac{\frac{\partial F}{\partial y}(2,-1,1)}{\frac{\partial F}{\partial z}(2,-1,1)} = -\frac{9}{7}.$$

6. By the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}e^r\cos\theta + \frac{\partial z}{\partial y}e^r\sin\theta.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -\frac{\partial z}{\partial x}e^r\sin\theta + \frac{\partial z}{\partial y}e^r\cos\theta.\end{aligned}$$

It follows that

$$\begin{aligned}e^{-2r}\left[\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right] &= e^{-2r}\left[\left(\frac{\partial z}{\partial x}e^r\cos\theta + \frac{\partial z}{\partial y}e^r\sin\theta\right)^2 + \left(-\frac{\partial z}{\partial x}e^r\sin\theta + \frac{\partial z}{\partial y}e^r\cos\theta\right)^2\right] \\ &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \left(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right)^2 \\ &= \left(\frac{\partial z}{\partial x}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.\end{aligned}$$

7. By the chain rule,

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial y}{\partial u}$$

Now

$$\frac{\partial u}{\partial x} = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

So,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \frac{dw}{du}\left(\frac{xy(y^2-x^2))}{(x^2+y^2)^2} + \frac{xy(x^2-y^2)}{(x^2+y^2)^2}\right) = 0.$$

8. By the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta.\end{aligned}$$

So,

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2 + \left(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right)^2 \\ &= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.\end{aligned}$$

9. (a) Since

$$x\sin y + xz^2 = 2e^{yz},$$

this means that

$$x = \frac{2e^{yz}}{\sin y + z^2}.$$

So if

$$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R},$$

is the function given by

$$f(y,z) = \frac{2e^{yz}}{\sin y + z^2},$$

then the surface we are interested in is the graph of this function. Note that

$$\nabla f = \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{2ze^{yz}(\sin y + z^2) - 2e^{yz}\cos y}{(\sin y + z^2)^2}, \frac{2ye^{yz}(\sin y + z^2) - 2e^{yz}2z}{(\sin y + z^2)^2}\right).$$

Using this, the equation of the tangent plane at the point $(y,z) = (\frac{\pi}{2}, 0)$ is

$$\begin{aligned}x &= f(\frac{\pi}{2}, 0) + \nabla f(\frac{\pi}{2}, 0)\cdot(y - \frac{\pi}{2}, z - 0) \\ &= 2 + (0, \pi)\cdot(y - \frac{\pi}{2}, z) \\ &= 2 + \pi z.\end{aligned}$$

(b) Let $g(x,y,z) = x\sin y + xz^2 - 2e^{yz}$. Then

$$\nabla g = (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) = (\sin y + z^2, x\cos y - 2ze^{yz}, 2xz - 2ye^{yz}).$$

At the point $(x,y,z) = (2, \frac{\pi}{2}, 0)$, we have

$$\nabla g(2, \frac{\pi}{2}, 0) = (1, 0, -\pi).$$

So the equation of the tangent plane at the point $(x,y,z) = (2, \frac{\pi}{2}, 0)$ is

$$(1, 0, -\pi)(x-2, y-\frac{\pi}{2}, z) = 0,$$

that is

$$x - 2 - \pi z = 0.$$

10. If $f(x,y,z) = 7x^2 - 12x - 5y^2 - z$, then

$$\nabla f = (14x - 12, -10y, -1),$$

so that a normal to the tangent plane is

$$\nabla f(2, 1, -1) = (16, -10, -1).$$

If $g(x,y,z) = xyz^2$, then

$$\nabla g = (yz^2, xz^2, 2xyz),$$

so that a normal to the tangent plane is

$$\nabla g = (1, 2, -4).$$

We check that these two vectors are orthogonal:

$$(16, -10, -1) \cdot (1, 2, -4) = 16 - 20 + 4 = 0,$$

so that the two tangent planes are indeed orthogonal.