MODEL ANSWERS TO HWK #4
(18.022 FALL 2010)

(1) (i) f is nowhere continuous. Let $\epsilon = \frac{1}{2}$. If x is rational, since irrational numbers are dense in reals, for any $\delta > 0$ there is irrational y such that $|y-x| < \delta$, and $|f(y)-f(x)| = 1 > \epsilon$. Therefore f is not continuous at x. If x is irrational, similarly from that rational numbers are dense in reals, for any $\delta > 0$ there is rational y such that $|y - x| < \delta$, and $|f(y) - f(x)| = 1 > \epsilon$. Therefore f is not continuous at x.

(ii) f is continuous only at $x = 0$. If $x \neq 0$, then let $\epsilon = \frac{|x|}{2}$. Now by the same argument as in (i), for any $\delta > 0$ there is y such that $|f(y) - f(x)| = |x| > \epsilon$. Therefore f is not continuous at $x \neq 0$. If $x = 0$, then for any $\epsilon > 0$, $|f(y) - f(x)| = |f(y)| < \epsilon$ for y such that $|y - x| = |y| < \frac{\epsilon}{2}$. Therefore f is continuous at $x = 0$.

(2) (i) If f is continuous at x then $\forall\epsilon > 0$, $\exists\delta > 0$ such that $\forall||y - x|| < \delta$, $||f(y) - f(x)|| < \epsilon$. Since $||f(y)-f(x)|| = \sqrt{(f_1(y) - f_1(x))^2 + \cdots + (f_m(y) - f_m(x))^2} > |f_i(y) - f_i(x)|$ for all $i = 1, \ldots, m$, f_i is also continuous at x for all i.

(ii) If f_i's are all continuous at x, then $\forall\epsilon > 0$ and $\forall i = 1, \ldots, m$, $\exists\delta_i > 0$ such that $\forall||y - x|| < \delta_i$, $|f_i(y) - f_i(x)| < \frac{\epsilon}{\sqrt{m}}$. Let $\delta = \min(\delta_1, \ldots, \delta_n)$. We get $\forall||y - x|| < \delta$, $||f(y) - f(x)|| = \sqrt{(f_1(y) - f_1(x))^2 + \cdots + (f_m(y) - f_m(x))^2} < \epsilon$. Therefore f is continuous at x.

(3) (a) By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0,$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

Therefore both partial derivatives exist at (0,0). Now

$$h(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = 0$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{||(x,y) - (0,0)||} = \lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{x^2 + y^2}} = \lim_{r\to 0} \frac{r^2|\sin\theta\cos\theta|}{r} = 0,$$

and $f(x, y)$ is differentiable at $(0, 0)$.

(b) By the symmetry between x and y, it's enough to prove the claim for $f_x(x, y)$. We have,

$$f_x(a,b) = |b| \lim_{h\to 0} \frac{|a+h| - |a|}{h} = \begin{cases} |b|, & a > 0 \\ -|b|, & a < 0 \end{cases}$$

So f_x is not continuous at $(0, b)$ if $|b| > 0$. For any neighborhood of the origin, we can choose such point $(0, b)$, hence f_x is not continuous in any neighborhood of the origin.

(4) Viewing f_y as a function of y, it's easy to find a function g such that $g_y = f_y$, namely

$$g(x,y) = x^3y^2 + x\cos(xy) + y^3$$

Now let $f(x,y) = g(x,y) + h(x) = x^3y^2 + x\cos(xy) + y^3 + h(x)$. We get

$$3x^2y^2 - xy\sin(xy) + \cos(xy) = f_x(x,y) = 3x^2y^2 + \cos(xy) - xy\sin(xy) + h'(x)$$

Therefore $h'(x) = 0$ and $h(x)$ must be a constant. So $f(x,y) = x^3y^2 + x\cos(xy) + y^3 + C$ for some constant C will do.

(5) (2.3.21) $D\mathbf{f}(x,y,z) = \begin{pmatrix} yz & xz & xy \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}$, therefore $D\mathbf{f}(\mathbf{a})$ is $\begin{pmatrix} 0 & -2 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \end{pmatrix}$

(6) (2.3.25) $D\mathbf{f}(s,t) = \begin{pmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{pmatrix}$, therefore $D\mathbf{f}(\mathbf{a})$ is $\begin{pmatrix} -2 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$

(7) (2.3.30) The plane is perpendicular to the gradient vector of $f(x,y,z) = z - 4\cos(xy)$ at $(\frac{\pi}{3}, 1, 2)$, which is $(2\sqrt{3}, \frac{2\sqrt{3}\pi}{3}, 1)$, and passes through the point $(\frac{\pi}{3}, 1, 2)$. Hence the equation of the plane is $(2\sqrt{3}, \frac{2\sqrt{3}\pi}{3}, 1)\cdot(x-\frac{\pi}{3}, y-1, z-2) = 0$. Rewriting it, we get $2\sqrt{3}x + \frac{2\sqrt{3}\pi}{3}y + z = 2 + \frac{4\sqrt{3}\pi}{3}$.

(8) (2.3.31) The plane is perpendicular to the gradient vector of $f(x,y,z) = z - \exp x + y\cos(xy)$ at $(0,1,e)$, which is $(-e,-e,1)$, and passes through the point $(0,1,e)$. Hence the equation of the plane is $(-e,-e,1)\cdot(x, y-1, z-e) = 0$. Rewriting it, we get $ex + ey - z = 0$.

(9) (2.3.33) $x_5 = -8 + (-2)\times 2(x_1-2) + (-6)\times(-1)(x_2+1) + (-4)\times 1(x_3-1) + (-2)\times 3(x_4-3) = -4x_1 + 6x_2 - 4x_3 - 6x_4 + 28$.

(10) (2.3.51) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$. Then $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$. Hence,

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = A$$

This agrees with the fact that the derivative of $f(x) = ax$ is the slope a, when A is 1×1 matrix.