MODEL ANSWERS TO HWK #3
(18.022 FALL 2010)

(1) In cartesian coordinates D is the region

$$\begin{cases} (x-a)^2+(y-a)^2 \leq a^2 \\ -1 \leq z \leq 3 \end{cases}$$

This translates to the cylindrical coordinates

$$\begin{cases} r^2-2ar(\cos\theta+\sin\theta) \leq a^2 \\ -1 \leq z \leq 3 \end{cases}$$

(2) The region D is

$$\begin{cases} \rho \leq 2a \\ -a \leq \rho\sin\phi\cos\theta \leq a \end{cases}$$

(3) (i) True. For any $c \in C$ there exists $b \in B$ such that $g(b)=c$, since g is surjective. Since f is surjective there exists $a \in A$ such that $f(a)=b$ and hence $(g \circ f)(a)=c$. It follows that $g \circ f$ is surjective.
(ii) False. Consider the counter example given by the domains $A=\{1\}$, $B=\{0,1\}$ and $C=\{1\}$ and the functions f,g defined by $f(1)=1$ and $g(0)=g(1)=1$. Then $g \circ f : A \to C$ is surjective (since $g(f(1))=1$), but f is not surjective since $0 \in B$ has no preimage.
(iii) True. For any $c \in C$ there exists $a \in A$ such that $g(f(a))=c$ since $g \circ f$ is surjective. Since $f(a) \in B$ we learn that g is surjective.

(4) We take f to be

$$f(x)=\begin{cases} c, & x \in S \\ c-1, & x \notin S \end{cases}$$

(5) (2.1.34)
(a) We take F to be

$$F(x,y,z)=\begin{cases} 1, & x^2+xy-xz=2 \\ 0, & x^2+xy-xz \neq 2 \end{cases}$$

(b) We take f to be

$$f(x,y)=\frac{x^2+xy-2}{x},$$

defined for any (x,y) such that $x \neq 0$.

(6) (2.2.9) The limit does not exist. Along the line $y=-x$ the limit is 0 and along the line $y=x$ the limit is 2.

(7) (2.2.11) The limit does not exist. Along the line $y = 0$ the limit is 2 and along the line $x = 0$ the limit is 1.

(8) (2.2.13) The limit exists and is 0.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy + y^2}{x + y} = \lim_{(x,y)\to(0,0)} \frac{(x + y)^2}{x + y} = \lim_{(x,y)\to(0,0)} x + y = 0\,.$$

(9) (2.2.15) The limit exists and is 0.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} x^2 - y^2 = 0\,.$$

(10) (2.2.31)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho\to 0} \frac{\rho^3 \sin^2\phi \cos\theta \sin\theta \cos\phi}{\rho^2} = 0\,.$$

(11) (2.2.35) The function is continuous because it is a polynomial.

(12) (2.2.42) The function g is clearly continuous at any point $(x, y) \neq (0, 0)$. So for g to continuous c must be the limit $\lim_{(x,y)\to(0,0)} g(x, y)$. We compute

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2}{x^2 + y^2} + 2 = \lim_{r\to 0} \frac{r^3(\cos^3\theta + \cos\theta\sin^2\theta)}{r^2} + 2 = 2\,,$$

so $c = 2$ is the value that makes g continuous.