MODEL ANSWERS TO HWK #2
(18.022 FALL 2010)

(1) (1.3.27)

(a) First let us fix a labeling for the points: W_1, W_2 and W_3 are the centers of what we will call the circles 1, 2 and 3, respectively, O is the point where all three circles intersect, and A, B and C are the points of pairwise intersections: circles 2 and 3 intersect at point A, circles 1 and 3 intersect at point B and circles 1 and 2 intersect at point C. Then,

$$\begin{aligned}\mathbf{a} &= \mathbf{w_2} + \mathbf{w_3}\\ \mathbf{b} &= \mathbf{w_1} + \mathbf{w_3}\\ \mathbf{c} &= \mathbf{w_1} + \mathbf{w_2}.\end{aligned}$$

(b) In order to show that A, B and C lie on a circle of radius r, we have to show that $\left\|\vec{AP}\right\| = \left\|\vec{BP}\right\| = \left\|\vec{CP}\right\| = r$, where P is the center of the circle (and as noted in the textbook, $\vec{OP} = \mathbf{w_1} + \mathbf{w_2} + \mathbf{w_3}$):

$$\begin{aligned}\vec{AP} &= \vec{AO} + \vec{OP} = -\mathbf{a} + \mathbf{w_1} + \mathbf{w_2} + \mathbf{w_3} = \mathbf{w_1}\\ \vec{BP} &= \vec{BO} + \vec{OP} = -\mathbf{b} + \mathbf{w_1} + \mathbf{w_2} + \mathbf{w_3} = \mathbf{w_2}\\ \vec{CP} &= \vec{CO} + \vec{OP} = -\mathbf{c} + \mathbf{w_1} + \mathbf{w_2} + \mathbf{w_3} = \mathbf{w_3}\end{aligned}$$

Since $\|\mathbf{w_1}\| = \|\mathbf{w_2}\| = \|\mathbf{w_3}\| = r$, we are done.

(c) In order to show that O is the common intersection point of the altitudes perpendicular to the edges, it is enough to show that $\vec{AB} \perp \vec{OC}$, $\vec{BC} \perp \vec{OA}$ and $\vec{CA} \perp \vec{OB}$:

$$\vec{AB} \cdot \vec{OC} = (\vec{AO} + \vec{OB}) \cdot \mathbf{c} = (-\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = (\mathbf{w_1} - \mathbf{w_2}) \cdot (\mathbf{w_1} + \mathbf{w_2}) = 0$$

The other two are similar to this one, so I won't write them out.

(2) The triple $\mathbf{u} = (1, -1, 2), \mathbf{v} = (2, 1, 1), \mathbf{w} = (0, 2, -1)$ is a right-hand triple because

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = 3,$$

which is positive.

(3) (a) Write $\mathbf{v} = (\|\mathbf{v}\| \cos\alpha, \|\mathbf{v}\| \sin\alpha) = \|\mathbf{v}\| (\cos\alpha, \sin\alpha)$ and let us compute $\mathbf{w} = A(\theta)\mathbf{v}$:

$$\begin{aligned}
\mathbf{w} &= \|\mathbf{v}\| \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \\
&= \|\mathbf{v}\| \begin{bmatrix} \cos\theta\cos\alpha + \sin\theta\sin\alpha \\ -\sin\theta\cos\alpha + \cos\theta\sin\alpha \end{bmatrix} \\
&= \|\mathbf{v}\| \begin{bmatrix} \cos(\alpha-\theta) \\ \sin(\alpha-\theta) \end{bmatrix}.
\end{aligned}$$

Thus $\mathbf{w} = (\|\mathbf{v}\|\cos(\alpha-\theta), \|\mathbf{v}\|\sin(\alpha-\theta))$, which is exactly the vector $\mathbf{v}$ rotated clockwise around the origin by an angle of θ. Hence, multiplying on the left by $A(\theta)$ corresponds to rotating clockwise around the origin by θ.

(b)

$$\begin{aligned}
BC &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & \cos\theta\sin\phi + \sin\theta\cos\phi \\ -\sin\theta\cos\phi - \cos\theta\sin\phi & -\sin\theta + \cos\theta\cos\phi \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\phi) & \sin(\theta+\phi) \\ -\sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}
\end{aligned}$$

(c) Applying BC to a vector $\mathbf{v}$ (to get $BC\mathbf{v}$) is the same as first applying C to $\mathbf{v}$ and then B to $C\mathbf{v}$. But this corresponds exactly to rotating $\mathbf{v}$ clockwise around the origin by ϕ and then by θ, which is the same as rotating by $\theta + \phi$.

(4) (1.4.6)

$$(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

(5) (1.4.11)

Note that $\vec{AB} = \vec{CD} = (3, -4, -2)$ and $\vec{AC} = \vec{BD} = (-4, -1, -3)$, so the parallelogram is $ABDC$, and its area is

$$\|(3,-4,-2) \times (-4,-1,-3)\| = \left\| \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -2 \\ -4 & -1 & -3 \end{bmatrix} \right\| = \|(10, 17, -19)\| = 5\sqrt{30}.$$

(6) (1.4.18)

To find the volume of the parallelepiped we use the triple scalar product:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = ((3,-1,0) \times (-2,0,1)) \cdot (1,-2,4) = (-1,-3,-2) \cdot (1,-2,4) = -3.$$

The volume of the parallelepiped is 3.

(7) (1.4.26)

(a),(f),(h) are vectors; (d),(g) are scalars; (b),(c),(e) do not make sense.

(8) (1.5.7)

A plane in $\mathbb{R}^3$ is determined uniquely by a point $P_0 = (x_0, y_0, z_0)$ on the plane and a vector $\mathbf{n}$ perpendicular to the plane. In this case a point $P_0 = (1, -1, 2)$ is given, and $\mathbf{n}$ must be parallel to the prescribed line, so take $\mathbf{n} = (3, -2, -1)$.

Then the equation for the plane is

$$\begin{aligned} &\mathbf{n} \cdot \vec{P_0P} = 0 \\ \iff &(3, -2, -1) \cdot (x-1, y+1, z-2) = 0 \\ \iff &3(x-1) - 2(y+1) - (z-2) = 0 \\ \iff &3x - 2y - z = 3. \end{aligned}$$

(9) (1.5.8)

A plane in $\mathbb{R}^3$ is determined uniquely by a point $P_0 = (x_0, y_0, z_0)$ on the plane and a vector $\mathbf{n}$ perpendicular to the plane.

- Finding P_0: Any point on the line l_1 or on the line l_2 works. For example, setting $t = 0$ on the equation for l_1, we get $P_0 = (2, -5, 1)$.
- Finding $\mathbf{n}$: The vector $\mathbf{n}$ must be perpendicular to the vectors $(1, 3, 5)$ and $(-1, 3, -2)$ (vectors parallel to l_1 and l_2 respectively), so we can take it to be their cross product:

$$\mathbf{n} = (1, 3, 5) \times (-1, 3, -2) = (-21, -3, 6).$$

Then the equation for the plane is

$$\begin{aligned} &\mathbf{n} \cdot \vec{P_0P} = 0 \\ \iff &(-21, -3, 6) \cdot (x-2, y+5, z-1) = 0 \\ \iff &-21(x-2) - 3(y+5) + 6(z-1) = 0 \\ \iff &7x + y - 2z = 7. \end{aligned}$$

(10) (1.5.9)

A plane is determined uniquely by a point $P_0 = (x_0, y_0, z_0)$ on the line and a vector $\mathbf{v}$ parallel to the line.

- Finding P_0: The point P_0 must be in both planes, and hence satisfy both $x+2y-3z = 5$ and $5x + 5y - z = 1$. If we chose to set $z_0 = 0$ we must then have $x_0 + 2y_0 = 5$ and $5x_0 + 5y_0 = 1$, which yields $P_0 = (-\frac{23}{5}, \frac{24}{5}, 0)$.
- Finding $\mathbf{v}$: The vector $\mathbf{v}$ must be perpendicular to the vectors $(1, 2, -3)$ and $(5, 5, -1)$ (vectors normal to each of the planes given), so we can take it to be their cross product:

$$\mathbf{n} = (1, 2, -3) \times (5, 5, -1) = (13, -14, -5).$$

Then the equation for the line is $(x, y, z) = P_0 + t\mathbf{v}$:

$$\begin{cases} x = -\frac{23}{5} + 13t \\ y = \frac{24}{5} - 14t \\ z = -5t \end{cases}$$

(11) (1.5.12)

Two planes Π_1 and Π_2 are perpendicular if and only if their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular, i.e., if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. In this case, $\mathbf{n}_1 = (A, -1, 1)$ and $\mathbf{n}_2 = (3A, A, -2)$:

$$(A, -1, 1) \cdot (3A, A, -2) = 3A^2 - A - 2 = 0 \iff A = 1 \text{ or } A = -\frac{2}{3}.$$

(12) (1.5.20)

In order to find the distance D between $P_0 = (1, -2, 3)$ and the line l we need an auxiliary point B on the line l; take for example $B = (-5, 3, 4)$ (obtained by setting $t = 0$ on the equations for l). Then $D = \left\| \vec{BP_0} - \text{proj}_{\mathbf{v}} \vec{BP_0} \right\|$, where $\mathbf{v} = (2, -1, 0)$ is a vector parallel to the line l:

$$D = \left\| (6, -5, -1) - \frac{(2, -1, 0) \cdot (6, -5, -1)}{(2, -1, 0) \cdot (2, -1, 0)} (2, -1, 0) \right\| = \left\| (-\frac{4}{5}, -\frac{8}{5}, -1) \right\| = \sqrt{\frac{21}{5}}.$$

(13) (1.5.24)

In order to find the distance D between the lines l_1 and l_2 we need two auxiliary points B_1 and B_2, on l_1 and l_2 respectively. We can take $B_1 = (-7, 1, 3)$ and $B_2 = (0, 2, 1)$ (obtained by setting $t = 0$ on the equations for l_1 and l_2). Now, $D = \left\| \text{proj}_{\mathbf{n}} \vec{B_1 B_2} \right\|$, where $\mathbf{n}$ is a vector perpendicular to both lines (and hence perpendicular to the planes Π_1 and Π_2 parallel to each other and that contain the lines l_1 and l_2 respectively. Figure 1.79 in page 45 of the textbook is helpful in understanding this argument):

- Finding $\mathbf{n}$: Take $\mathbf{v_1} = (1, 5, -2)$ and $\mathbf{v_2} = (4, -1, 8)$ vectors parallel to the lines l_1 and l_2 respectively. Then we can take $\mathbf{n} = (1, 5, -2) \times (4, -1, 8) = (38, -16, -21)$.

Then,

$$D = \left\| \frac{\mathbf{n} \cdot \vec{B_1 B_2}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right\| = \left\| (\frac{11096}{2141}, -\frac{4672}{2141}, -\frac{6132}{2141}) \right\| = \frac{292}{\sqrt{2141}}.$$

(14) (1.5.28)

In order to find the distance D between the planes Π_1 and Π_2 we need two auxiliary points B_1 and B_2, on Π_1 and Π_2 respectively. We can take $P_1 = (0, 0, 6)$ and $B_2 = (0, 0, -2)$. Now, $D = \left\| \text{proj}_{\mathbf{n}} \vec{P_1 P_2} \right\|$, where $\mathbf{n}$ is a vector perpendicular to both planes (Π_1 and Π_2 are parallel!), so take $\mathbf{n} = (5, -2, 2)$ (note that $(-10, 4, -4) = -2(5, -2, 2)$). Then:

$$D = \left\| \frac{\mathbf{n} \cdot \vec{P_1 p_2}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right\| = \left\| (\frac{80}{33}, -\frac{32}{33}, \frac{32}{33}) \right\| = \frac{16}{\sqrt{33}}.$$

(15) (1.6.9)

$$\|\mathbf{a} - \mathbf{b}\| = \|(\mathbf{a} - \mathbf{c}) + (\mathbf{c} - \mathbf{b})\| \leq \|(\mathbf{a} - \mathbf{c})\| + \|(\mathbf{c} - \mathbf{b})\|.$$

In the last step we used the triangle inequality.

(16) (1.6.11)

Assume that $\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\|$. Then:

$$\begin{aligned} &\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a} - \mathbf{b}\|^2 \\ \iff &(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ \iff &\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ \iff &2\mathbf{a} \cdot \mathbf{b} = -2\mathbf{a} \cdot \mathbf{b} \\ \iff &\mathbf{a} \cdot \mathbf{b} = 0 \\ \iff &\mathbf{a} \perp \mathbf{b}. \end{aligned}$$

(17) (1.6.14)
Half of my inventory is worth $10 \times \$8 + 15 \times \$10 + 12 \times \$12 + 10 \times \$15 = \$524$.
Half of my friend's inventory is worth $15 \times \$10 + 8 \times \$10 + 10 \times \$12 + 14 \times \$15 = \$560$.
My friend is not likely to accept my offer, because he would lose the equivalent of \$36.

(18) (1.6.21)
We expand the first row:

$$\begin{aligned}\begin{vmatrix} 7 & 0 & -1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & -3 & 0 & 2 \\ 0 & 5 & 1 & -2 \end{vmatrix} &= (-1)^{1+1}(7)\begin{vmatrix} 0 & 1 & 3 \\ -3 & 0 & 2 \\ 5 & 1 & -2 \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 2 \\ 0 & 5 & -2 \end{vmatrix} \\ &= (7)(-5) + (-1)(7) \\ &= -42\end{aligned}$$