MODEL ANSWERS TO HWK #11
(18.022 FALL 2010)

(1) (6.1.1)

(a) $x' = (-3, 4)$ so $||x'|| = 5$, hence

$$\int_x f ds = 5\int_0^2 (2 - 3t + 8t - 2)dt = 50\,.$$

(b) $x' = (-\sin t, \cos t)$ so $||x'|| = 1$, hence

$$\int_x f ds = \int_0^\pi (\cos t + 2\sin t) = 4\,.$$

(2) (6.1.3) $x' = (1, 1, 3\sqrt{t}/2)$ so $||x'|| = \sqrt{2 + 9t/4}$, hence

$$\int_x f ds = \int_1^3 \frac{t + t^{3/2}}{t + t^{3/2}}\sqrt{2 + 9t/4}dt\,.$$

We perform the substitution $t = \frac{4}{9}(u - 2)$ and get that this equals

$$\int_{17/4}^{35/4} 4\sqrt{u}/9du = \frac{1}{27}[17^{3/2} - 35^{3/2}]\,.$$

(3) (6.1.7) $x' = (\cos t, \sin t)$, hence

$$\int_x F \cdot ds = \int_0^{\pi/2} [(-\cos t + 2)\cos t + \sin^2 t]dt\,.$$

It easy to see by change of variable that $\int_0^{\pi/2} \sin^2 t = \int_0^{pi/2} \cos^2 t$ and so the above intergal equals

$$\int_0^{\pi/2} 2\cos t dt = 2\,.$$

(4) (6.1.11) $x' = (-3\sin 3t, 3\cos 3t)$, hence

$$\int_x xdy - ydx = \int_0^\pi 3\cos^2 3t + 3\sin^2 3t = 3\pi\,.$$

(5) (6.1.13) $x' = (2e^{2t}\cos 3t - 3e^{2t}\sin 3t, 2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$, hence

$$\int_x \frac{xdx + ydy}{(x^2 + y^2)^{3/2}} = \int_0^{2\pi} \frac{2e^{4t}}{e^{6t}} = 1 - e^{-4\pi}\,.$$

(6) (6.1.16) A parametrization of the curve (there is more than one) is $x(t) = (t, 5 - 4t, 2t - 1)$ for $1 \le t \le 2$. We have $x' = (1, -4, 2)$, hence the work is

$$\int_1^2 [t^2(5 - 4t) - 4(2t - 1) + 2(6t - 5)]dt = \int_1^2 [-4t^3 + 5t^2 + 4t - 6]dt = -3\frac{1}{3}\,.$$

(7) (6.1.19) Parameterize C by a curve x defined by

$$x(t) = \begin{cases} (t,t) & 0 \le t \le 1\,, \\ (t,1) & 1 \le t \le 3\,, \\ (3,4-t) & 3 \le t \le 4\,, \\ (7-t,0) & 4 \le t \le 7\,. \end{cases}$$

Note that this curve have **clockwise** orientation, so we will remember to take -1 to whatever we get in the integral. We have

$$dx = \begin{cases} 1 & 0 \le t \le 3\,, \\ 0 & 3 \le t \le 4\,, \\ -1 & 4 \le t \le 7\,, \end{cases}$$

and

$$dy = \begin{cases} 1 & 0 \le t \le 1\,, \\ 0 & 1 \le t \le 3\,, \\ -1 & 3 \le t \le 4\,, \\ 0 & 4 \le t \le 7\,. \end{cases}$$

Thus

$$\int_C x^2y dx - (x+y)dy = -\int_0^1 t^3 dt - \int_1^3 t^2 dt + \int_0^1 2t dt - \int_3^4 (7-t)dt = -\frac{137}{12}\,.$$

(8) (6.1.21) A parametrization of the curve is $x(t) = (1+4t, 1+2t, 2-t)$ for $0 \le t \le 1$, so $x' = (4,2,-1)$. Hence

$$\int_C yzdx - xydy + xydz = \int_0^1 [4(1+2t)(2-t) - 2(1+4t)(2-t) - (1+4t)(1+2t)]dt = -\frac{11}{3}\,.$$

(9) (6.2.8) Let D be the ellipse. We have $N_x = 1, M_y = -4$, so by Green's theorem the work equals

$$\int\int_D -3dxdy = -12\pi\,.$$

(10) (6.2.10) Let $F(x,y) = (0,x)$ so that $N_x - M_y = 1$. Then by Green's theorem, the area is $-\int_0^{2\pi} a^2(t-\sin t)\sin t dt = 3\pi a^2$. (Since F is 0 on the x-axis)

(11) (6.2.11) C is negatively oriented, and $N_x - M_y = 5$. So by Green's theorem, the integral is just $-5 \times \text{Area} = -45$.

(12) (6.2.14) We need to subtract the area of the ellipse from 25π. Take $F(x,y) = (0,x)$ so that $N_x - M_y = 1$. Then by Green's theorem, the area of the ellipse is the line integral of F on the boundary of the ellipse. Let $(x,y) = (3\cos t, 2\sin t)$. The area of the ellipse is $\int_0^{2\pi} 6\cos^2 t dt = 6\pi$. Hence the area between the circle and the ellipse is 19π.

(13) (6.2.19) The integrand vector field is smooth everywhere. Since $N_x - M_y = 3x^2 - 3x^2 = 0$, by Green's theorem, the integral is 0.

(14) (6.2.20) The integrand vector field is smooth everywhere. Since $N_x - M_y = 3x^2+2+3y^2 > 0$ for all x,y, by Green's theorem, the integral has the same value as the double integral of a positive function. Hence it's always positive.

(15) (6.2.25) Let $F = \nabla f$. Then by the divergence theorem in the plane, we have

$$\oint_{\partial D} \nabla f \cdot \mathbf{n} ds = \int\int_D \nabla^2 f dA$$