MODEL ANSWERS TO HWK #10
(18.022 FALL 2010)

(1) We want to show that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$. We show that $I = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ by proving that

$$I^2 = \int_0^{\infty} e^{-x^2}dx \int_0^{\infty} e^{-y^2}dy = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy = \frac{\pi}{4}.$$

(i) The square of side a defined by $0 < x < a$ and $0 < y < a$ contains Q_a, the quarter circle of radius a, and thus:

$$\int_0^a\int_0^a e^{-(x^2+y^2)}dxdy \geq \iint_{Q_a} e^{-(x^2+y^2)}dxdy = \int_0^{\frac{\pi}{2}}\int_0^a e^{-r^2}rdrd\theta = \frac{\pi}{2}\left(\frac{-e^{-r^2}}{2}\bigg|_{r=0}^{a}\right) = \frac{\pi}{4}(1-e^{-a^2})$$

Taking the limit as a goes to ∞, we conclude that $I^2 \geq \frac{\pi}{4}$.

(ii) The square of side a defined by $0 < x < a$ and $0 < y < a$ is contained in $Q_{\sqrt{2}a}$, the quarter circle of radius $\sqrt{2}a$, and thus:

$$\int_0^a\int_0^a e^{-(x^2+y^2)}dxdy \leq \iint_{Q_{\sqrt{2}a}} e^{-(x^2+y^2)}dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}a} e^{-r^2}rdrd\theta = \frac{\pi}{2}\left(\frac{-e^{-r^2}}{2}\bigg|_{r=0}^{\sqrt{2}a}\right) = \frac{\pi}{4}(1-e^{-2a^2})$$

Taking the limit as a goes to ∞, we conclude that $I^2 \leq \frac{\pi}{4}$.

(2) (5.5.15)

$$\iint_D (x^2+y^2)^{3/2}dA = \int_0^{2\pi}\int_0^3 r^3rdrd\theta = 2\pi\left(\frac{r^5}{5}\bigg|_{r=0}^{3}\right) = \frac{486}{5}\pi$$

(3) (5.5.16)

$$\int_{-a}^a\int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2}dxdx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^a e^{r^2}rdrd\theta = \pi\left(\frac{e^{r^2}}{2}\bigg|_{r=0}^{a}\right) = \frac{\pi}{2}(e^{a^2}-1)$$

(4) (5.5.20)

$$\int_{\text{rose}} dA = 4\int_{\text{leaf}} dA = 4\int_0^{\frac{\pi}{2}}\int_0^{\sin 2\theta} rdrd\theta = 4\int_0^{\frac{\pi}{2}} \frac{(\sin 2\theta)^2}{2}d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}d\theta = \frac{\pi}{2} - \frac{1}{4}\left(\sin 4\theta\right|_{\theta=0}^{\frac{\pi}{2}} = \frac{\pi}{2}$$

(5) (5.5.21) The cardioid and the circle intersect at the points $(0,1)$ and $(0,-1)$, and since we want the area *inside* the cardioid and *outside* the circle, the bounds of integration for θ must

be $\pi/2$ to $3\pi/2$.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{1}^{1-\cos\theta} r dr d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}((1-\cos\theta)^2-1^2)d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\cos^2\theta-2\cos\theta)d\theta$$
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\frac{1+\cos 2\theta}{2}-2\cos\theta)d\theta$$
$$= \frac{\pi}{4}+\frac{1}{8}(\sin 3\pi-\sin\pi)-(sin\frac{3\pi}{2}-\sin\frac{\pi}{2}) = \frac{\pi}{4}+2$$

(6) (5.5.22)

$$\int_0^{2\pi}\int_0^{3\theta} r dr d\theta = \int_0^{2\pi}\frac{9\theta^2}{2}d\theta = \frac{9}{2}\left(\frac{\theta^3}{3}\right|_{\theta=0}^{2\pi} = \frac{9}{2}\,\frac{8\pi^3}{3} = 12\pi^3$$

(7) (5.5.29)

$$\iiint_W dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r dz dr d\theta = 2\pi\int_0^1 2\sqrt{10-2r^2}\,r dr$$
$$= -\pi\left(\frac{(10-2r^2)^{3/2}}{3/2}\right|_{r=0}^{1} = \frac{4\sqrt{2}}{3}\pi(5\sqrt{5}-8)$$

(8) (5.5.30)

$$\iiint_W dV = \int_0^{2\pi}\int_0^2\int_0^{9-r^2} r dz dr d\theta = 2\pi\int_0^2(9r-r^3)dr = 2\pi\left(\frac{9r^2}{2}-\frac{r^4}{4}\right|_{r=0}^{2} = 28\pi$$

(9) (5.5.31)

$$\iiint_W (2+x^2+y^2)dV = \int_0^{2\pi}\int_3^5\int_0^{\sqrt{25-z^2}} r dz dr d\theta = 2\pi\int_3^5\left(r^2+\frac{r^4}{4}\right|_{r=0}^{\sqrt{25-z^2}} dz$$
$$= \int_3^5(25-z^2+\frac{625-50z^2+z^4}{4})dz = \int_3^5(\frac{725}{4}-\frac{23}{2}z^2+\frac{z^4}{4})dz$$
$$= \frac{656}{5}\pi$$

(10) (5.5.32) By symmetry reasons, the total volume of the solid is 16 times the volume of the portion defined by $z>0$, $x>0$ and $0<y<x$: this sixteenth of the solid is bounded on the bottom by the plane $z=0$, on the sides by the planes $y=0$ and $y=x$ and by the cylinder $x^2+y^2=a^2$, and on top by $x^2+z^2=a^2$. We write the integral in cylindrical coordinates:

$$
\begin{aligned}
\text{Volume} &= 16\int_0^{\frac{\pi}{4}}\int_0^a\int_0^{\sqrt{a^2-r^2\cos^2\theta}} r\,dz\,dr\,d\theta \\
&= 16\int_0^{\frac{\pi}{4}}\int_0^a \sqrt{a^2-r^2\cos^2\theta}\,r\,dr\,d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{4}}\left(-\frac{(a^2-r^2\cos^2\theta)^{\frac{3}{2}}}{\cos^2\theta}\bigg|_{r=0}^{a}\right)d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{4}}\frac{1-\sin^3\theta}{\cos^2\theta}d\theta \\
&= \frac{16}{3}\left(\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}d\theta-\int_0^{\frac{\pi}{4}}\frac{\sin^3\theta}{\cos^2\theta}d\theta\right) \\
&= \frac{16}{3}\left((\tan\theta|_{\theta=0}^{\frac{\pi}{4}}+\int_{\frac{\sqrt{2}}{2}}^1\frac{t^2-1}{t^2}dt\right) \\
&= \frac{16}{3}\left(1+\int_{\frac{\sqrt{2}}{2}}^1 1-\frac{1}{t^2}dt\right) \\
&= \frac{16}{3}\left(1+(t|_{t=\frac{\sqrt{2}}{2}}^1+\left(\frac{1}{t}\bigg|_{t=\frac{\sqrt{2}}{2}}^1\right)\right) \\
&= \frac{16}{3}\left(1+1-\frac{\sqrt{2}}{2}+1-\sqrt{2}\right) \\
&= (16-8\sqrt{2})a^3
\end{aligned}
$$