MODEL ANSWERS TO HWK #1

1.1.20. (a) $\vec{0}$. Suppose that $\vec{a} = (a_1, a_2, a_3)$. Then

$$0 \cdot \vec{a} = (0 \cdot a_1, 0 \cdot a_2, 0 \cdot a_3) = (0, 0, 0) = \vec{0}.$$

The case of a vector in $\mathbb{R}^2$ is similar (and easier).
(b) $\vec{a}$. Suppose that $\vec{a} = (a_1, a_2, a_3)$. Then

$$1 \cdot \vec{a} = (1 \cdot a_1, 1 \cdot a_2, 1 \cdot a_3) = (a_1, a_2, a_3) = \vec{a}.$$

The case of a vector in $\mathbb{R}^2$ is similar (and easier).
1.1.22. Let $P_0 = (x_0, y_0, z_0)$ and let $P = (x, y, z)$ be a general point of the parallelogram. Then

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}.$$

Now

$$\overrightarrow{P_0P} = \lambda\vec{a} + \mu\vec{b},$$

where $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$. (Indeed to get to P from P_0, one slides along the side parallel to $\vec{a}$ and then slides in the direction of $\vec{b}$.) So

$$\begin{aligned}(x, y, z) &= (x_0, y_0, z_0) + \lambda(a_1, a_2, a_3) + \mu(b_1, b_2, b_3) \\ &= (x_0 + \lambda a_1 + \mu b_1, y_0 + \lambda a_2 + \mu b_2, z_0 + \lambda a_3 + \mu b_3).\end{aligned}$$

1.1.24. (a) 4 mph.
(b) Since $(5, 10) = 1/10(50, 100)$ it takes six minutes until the plane is directly above the skyscraper.
(c) In six minutes the plane climbs 2/5th of a mile. Now one mile is $5,280$ feet (google is your friend), so the plane is 2112 feet above the ground. So it clears the skyscraper by

$$2112 - 1250 = 862,$$

feet.
1.2.3. $(3, \pi, -7) = 3\hat{\imath} + \pi\hat{\jmath} - 7\hat{k}$.
1.2.10 $\pi\hat{\imath} - \hat{\jmath} = (\pi, -1, 0)$.
1.2.11 (a) We want c_1 and c_2 such that

$$(3, 1) = (c_1 + c_2, c_1 - c_2),$$

that is, we want

$$\begin{aligned}c_1 + c_2 &= 3 \\ c_1 - c_2 &= 1.\end{aligned}$$

Adding both equations we get $2c_1 = 4$, so that $c_1 = 2$ and subtracting both equations gives $2c_2 = 2$, so that $c_2 = 1$. It is easy to check that these values for c_1 and c_2 work.
(b) We want c_1 and c_2 such that

$$(3, -5) = (c_1 + c_2, c_1 - c_2),$$

that is, we want

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 - c_2 &= -5. \end{aligned}$$

Adding both equations we get $2c_1 = -2$, so that $c_1 = -1$ and subtracting both equations gives $2c_2 = 8$, so that $c_2 = 4$. It is easy to check that these values for c_1 and c_2 work.
(c) We want c_1 and c_2 such that

$$(b_1, b_2) = (c_1 + c_2, c_1 - c_2),$$

that is, we want

$$\begin{aligned} c_1 + c_2 &= b_1 \\ c_1 - c_2 &= b_2. \end{aligned}$$

Adding both equations we get $2c_1 = b_1 + b_2$, so that $c_1 = (b_1 + b_2)/2$ and subtracting both equations gives $2c_2 = b_1 - b_2$, so that $c_2 = (b_1 - b_2)/2$. We check that these values actually work:

$$\frac{b_1 + b_2}{2}(1, 1) + \frac{b_1 - b_2}{2}(1, -1) = (b_1, b_2) = \vec{b},$$

as expected and required.
1.2.14. $(x, y, z) = (12, -2, 0) + t(5, -12, 1) = (12 + 5t, -2 - 12t, t)$.
1.2.16 $(x, y, z) = (2, 1, 2) + t(3 - 2, -1 - 1, 5 - 2) = (2 + t, 1 - 2t, 2 + 3t)$.
1.2.24. If we plug in $t = 0$, then we see that the point $(-5, 2, 1)$ lies on the first line. Now if this point lies on the second line, then we may find t such that $(1 - 2t, 11 - 3t, 6t - 17) = (-5, 2, 1)$. Comparing the first coordinates, we see that $1 - 2t = -5$, that is, $t = 3$. It is easy to see that then the second and third coordinates agree as well. So the two lines share the point $(-5, 2, 1)$.
If we plug in $t = 0$ to the second line, then we get the point $(1, 11, -17)$. Now if this point lies on the first line, then we may find t such that $(2t - 5, 3t + 2, 1 - 6t) = (1, 11, -17)$. Looking at the first coordinate, we must have $t = 3$, and then it is easy to see that the second and third coordinates come out right.
So the two lines share two points. As any two points determine a unique line, l_1 and l_2 must indeed be the same line.
1.2.28. (a) $(7, -2, 1)$ and $(13, 1, -8)$.

(b) $(2, 1, -3)$.
(c) When $t - 2 = 1/6$, so that $t = 2 + 1/6$, that is, after 130 seconds.
(d) The bird has y-coordinate equal to 4 after exactly six minutes. At this point its x-coordinate is 19, so no, the bird is never at this point (assuming the bird does not cheat and change directions on us).
1.2.30. We want to find t such that

$$5(1 - 4t) - 2(t - 3/2) + (2t + 1) = 1,$$

so that

$$-20t + 9 = 1,$$

and so $t = -2/5$. This is the point $(1 - 8/5, -2/5 - 3/2, -4/5 + 1) = (-3/5, -21/10, -1/5)$.
1.2.35. We want to know if we can find s and t such that

$$(2s + 1, -3s, s - 1) = (3t + 1, t + 5, 7 - t).$$

Adding the last two coordinates, we get $-2s-1 = 12$, so that $s = 13/2$. Adding all of the coordinates together, we get $0 = 3t + 13$, so that $t = -13/3$. But then the third coordinate is a fraction with denominator 2, looking at the LHS and with denominator 3, looking at the RHS. As this is absurd, there are no such s and t and so the lines don't intersect.
1.2.38. Let $P = (x, y)$. The point A has coordinates (at, a) at time t. We have

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}.$$

Relative to A, the point P traces a circle, clockwise (anti, anti-clockwise, as it were), starting at the point $(0, -b)$. In other words the angle is $3\pi/2 - t$, at time t and so

$$\overrightarrow{AP} = (b\cos(3\pi/2 - t), b\sin(3\pi/2 - t)) = (-b\sin t, -b\cos t).$$

Therefore

$$(x, y) = \overrightarrow{OP} = (at - b\sin t, a - b\cos t).$$

There are machines, much like lawnmowers, whose job it is to make holes in the lawn. They have spikes instead of blades. If the point P is the endpoint of one of the spikes, then this is an example where $b > a$.
1.3.4. $\vec{a}\cdot\vec{b} = 2-2+0 = 0$, $\|\vec{a}\| = \sqrt{4+1} = \sqrt{5}$, and $\|\vec{b}\| = \sqrt{1+4+9} = \sqrt{14}$.
1.3.8. $\vec{a} \cdot \vec{b} = -1 + 2 - 2 = -1$, $\|\vec{a}\| = \sqrt{3}$ and $\|\vec{b}\| = 3$. It follows that

$$-1 = 3\sqrt{3}\cos\theta \qquad \text{so that} \qquad \cos\theta = \frac{-1}{3\sqrt{3}}.$$

So $\pi/2 < \theta < \pi$. In fact

$$\theta \approx 1.764.$$

1.3.12. $\vec{a} \cdot \vec{b} = 2 - 4 + 2 = 0$. So $\operatorname{proj}_{\vec{a}} \vec{b} = \vec{0}$.

1.3.13. Let $\vec{v} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$. Then

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}(2\hat{\imath} - \hat{\jmath} + \hat{k}),$$

is a unit vector which points in the direction of $\vec{v}$.

13.1.17. We suppose that neither $\vec{a}$ nor $\vec{b}$ is the zero vector.

$\operatorname{proj}_{\vec{a}} \vec{b} = \operatorname{proj}_{\vec{b}} \vec{a}$ if and only if either $\vec{a}$ and $\vec{b}$ are orthogonal or $\vec{a} = \vec{b}$.

If $\vec{a}$ and $\vec{b}$ are orthogonal, then both projections are the zero vector. If $\vec{a} = \vec{b}$ then both projections are equal to $\vec{a}$. So one direction is clear.

Suppose that $\operatorname{proj}_{\vec{a}} \vec{b} = \operatorname{proj}_{\vec{b}} \vec{a}$. If $\vec{a}$ and $\vec{b}$ are not orthogonal, then both sides of this equation are non-zero vectors. As the LHS is parallel to $\vec{a}$ and the RHS is parallel to $\vec{b}$, it follows that $\vec{a}$ and $\vec{b}$ are parallel. In this case the LHS is equal to $\vec{b}$ and the RHS is equal to $\vec{a}$. But then $\vec{a} = \vec{b}$.