

THIRD MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature: _____

Recitation Time: _____

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) For what values of λ does the function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$f(x, y, z) = \lambda x^2 - \lambda xy + y^2 + \lambda z^2,$$

have a non-degenerate local minimum at $(0, 0, 0)$?

$$Df = (2\lambda x - \lambda y, 2y - \lambda x, 2\lambda z)$$

$$Hf = \begin{pmatrix} 2\lambda & -\lambda & 0 \\ -\lambda & 2 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$$

$$d_1 = 2\lambda, \quad d_2 = 4\lambda - \lambda^2, \quad d_3 = 2\lambda$$

Minimum: $d_1 > 0, d_2 > 0, d_3 > 0$

$$\text{So } \lambda > 0, \quad 4\lambda - \lambda^2 > 0, \quad \lambda > 0.$$

$$\lambda(4 - \lambda) > 0$$

\Downarrow

$$\lambda < 4.$$

$$0 < \lambda < 4. \quad \lambda \in (0, 4).$$

2. (20pts) Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be the function $f(x, y, z) = x^2 - y^2 + z^2$.

(i) Show that f has a global maximum on the ellipsoid $2x^2 + 3y^2 + z^2 = 6$.

$K = \{(x, y, z) \in \mathbb{R}^3 \mid 2x^2 + 3y^2 + z^2 = 6\}$ is closed + bounded.

So K is compact.

f is cts, K is compact $\Rightarrow f$ has a global maximum.

(ii) Find this maximum.

Consider $h: \mathbb{R}^4 \rightarrow \mathbb{R}$ given by
 $h(x, y, z, \lambda) = x^2 - y^2 + z^2 + \lambda(2x^2 + 3y^2 + z^2 - 6)$.

Critical pts of h :
 $2x = -4\lambda x$
 $2y = 6\lambda y$
 $2z = -2\lambda z$
 $2x^2 + 3y^2 + z^2 = 6$.

Either $x=y=0, \lambda=1; y=z=0, \lambda=-\frac{1}{2}; x=z=0, \lambda=\frac{1}{3}$

$x=y=0, z=\sqrt{6}; y=z=0, x=\sqrt{3}; x=z, y=\sqrt{2}$

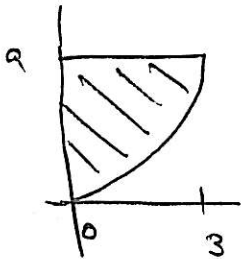
Of these three pts, ~~second~~ ^{first} gives biggest pt.

Absolute maximum is $\textcircled{6}$.

3. (20pts)

(i) Switch the order of integration in the integral

$$\int_0^3 \int_{x^2}^9 x e^{-y^2} dy dx.$$



$$\int_0^9 \int_0^{\sqrt{y}} x e^{-y^2} dx dy$$

(ii) Evaluate this integral.

$$\int_0^9 e^{-y^2} \left(\int_0^{\sqrt{y}} x dx \right) dy = \int_0^9 e^{-y^2} \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy$$

$$= \frac{-1}{4} \int_0^9 2ye^{-y^2} dy$$

$$= \frac{-1}{4} \left[e^{-y^2} \right]_0^9$$

$$= \frac{1}{4} (1 - e^{-81})$$

4. (20pts) Let W be the region inside the sphere $x^2 + y^2 + z^2 = 1$ and inside the cone $z^2 = x^2 + y^2$.

Set up an integral to calculate the integral of the function yz over W and calculate this integral.

View W as a region of type 1.

$$\iiint_W yz \, dz \, dy \, dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(\int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} y \left(\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z \, dz \right) dy \right) dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(\int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} y \left(\frac{1-x^2-y^2-x^2-y^2}{2} \right) dy \right) dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 0 \, dx = 0$$

as y, y^3 are odd functions.

In retrospect $J(y) = \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z \, dz$ is an even function y , so that $yJ(y)$ is an odd function.

Therefore integral is zero.

5. (20pts) Let D be the region in the first quadrant bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $xy = 1$ and $xy = 3$.

(i) Find $du dv$ in terms of $dx dy$, where $u = x^2 - y^2$ and $v = xy$.

$$\frac{\partial(u,v)}{\partial(x,y)}(u,v) = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2) \quad \frac{\partial(x,y)}{\partial(u,v)}(x,y) = \frac{1}{2(x^2 + y^2)}$$

$$du dv = 2(x^2 + y^2) dx dy$$

(ii) Evaluate the integral

$$\iint_D (x^4 - y^4) dx dy.$$

$$\int_1^3 \int_1^4 \frac{u}{2} du dv = \frac{1}{4} \int_1^3 [u^2]_1^4 dv$$

$$= \frac{1}{4} \int_1^3 15 dv$$

$$= \frac{15}{2}$$