SECOND MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function given by $f(x, y, z) = x^3y + y^3z + z^3x - 2xyz$.

(i) Find the gradient of f at $P = (2, -1, 1)$.

$$\nabla f(x,y,z) = (3x^2y + z^3 - 2yz)\hat{\imath} + (x^3 + 3y^2z - 2xz)\hat{\jmath} + (y^3 + 3z^2x - 2xy)\hat{k}$$

$$\nabla f(2,-1,1) = (-12 + 1 + 2)\hat{\imath} + (8 + 3 - 2\cdot 2)\hat{\jmath} + (-1 + 6 + 4)\hat{k}$$

$$= -9\hat{\imath} + 7\hat{\jmath} + 9\hat{k}$$

(ii) Find the directional derivative of f at P in the direction of $\hat{u} = \frac{2}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$.

$$D_{\hat{u}}f(2,-1,1) = \nabla f(P)\cdot\hat{u} = (-9,7,9)\cdot\left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right)$$

$$= -6 - \frac{14}{3} + 3$$

$$= \frac{-23}{3}$$

(iii) Find the tangent plane at the point P of the level surface

$$\{\, Q \in \mathbb{R}^3 \mid f(Q) = -3 \,\}.$$

$$\nabla f(P)\cdot\overrightarrow{PQ} = 0 \qquad (-9,7,9)\cdot(x-2, y+1, z-1) = 0$$

$$-9(x-2) + 7(y+1) + 9(z-1) = 0.$$

2. (20pts) Suppose that $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is differentiable at $P = (-1, 4)$ with derivative

$$DF(-1,4) = \begin{pmatrix} -1 & 1 \\ 3 & -2 \\ -2 & -1 \end{pmatrix}.$$

Suppose that $F(-1, 4) = (1, -1, 3)$. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(x, y) = \|F(x, y)\|$.

(i) Show that the function $f(x, y)$ is differentiable at P.

Let $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $g(u,v,w) = \|(u,v,w)\|$
$= (u^2+v^2+w^2)^{1/2}$

Then g is differentiable, and f is the composition of F and g, $f = g \circ F : \mathbb{R}^2 \longrightarrow \mathbb{R}$.
So f is differentiable at $(-1,4)$.

(ii) Find $Df(-1, 4)$.

$Df = Dg \cdot DF \qquad Dg = \frac{1}{(u^2+v^2+w^2)^{1/2}}(u,v,w)$

$Dg(1,-1,3) = \frac{1}{\sqrt{11}}(1,-1,3)$

$$Df(-1,4) = Dg(1,-1,3)\cdot DF(-1,4)$$
$$= \frac{1}{\sqrt{11}}(1,-1,3)\begin{pmatrix} -1 & 1 \\ 3 & -2 \\ -2 & -1 \end{pmatrix} = \frac{1}{\sqrt{11}}(-10, 0).$$

3. (20pts) Let

$$S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^3y + y^2z^3 + zx^2 = 3 \}.$$

(i) Show that S is the graph of a function $z = f(x, y)$ in a neighbourhood of $P = (1, -2, 1)$.

Let $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $F(x,y,z) = x^3y + y^2z^3 + zx^2 - 3$.

$DF = \nabla F = (3x^2y + 2xz,\ x^3 + 2yz^3,\ 3y^2z^2 + x^2)$

$DF(1,-2,1) = (-4, -3, 13)$.

As the last entry is non-zero, the implicity function theorem implies that f exists.

(ii) Find the partial derivatives

$$\frac{\partial f}{\partial x}(1, -2) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(1, -2).$$

Let $G: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $G(x,y) = F(x,y,f(x,y))$

Then G is identically zero. So $DG = \nabla G = (0,0)$

<u>Method I</u> $\quad 0 = \frac{\partial G}{\partial x} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial x}$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial x}$$

Plug in $(1,-2)$ $\quad 0 = -4 + 13 \cdot \frac{\partial f}{\partial x}(1,-2)$ $\quad \frac{\partial f}{\partial x}(1,-2) = \frac{4}{13}$.

Similarly $\frac{\partial f}{\partial y}(1,-2) = \frac{3}{13}$.

Method II Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the function

$$g(x,y) = (x, y, f(x,y)).$$

Then G is the composition of g and F, $G = F \circ g$

$$DG = DF \cdot Dg \qquad Dg = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{pmatrix}$$

$$(0,0) = DG(1,-2) = DF(1,-2,1) \cdot Dg(1,-2)$$

$$= (-4, -3, 13) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_x(1,-2) & f_y(1,-2) \end{pmatrix}$$

$$= \left(-4 + 13 f_x(1,-2), \; -3 + 13 f_y(1,-2)\right)$$

So $f_x(1,-2) = \dfrac{4}{13}$, $f_y(1,-2) = \dfrac{3}{13}$.

4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{N}(a) = \frac{2}{7}\hat{\imath} - \frac{6}{7}\hat{\jmath} - \frac{3}{7}\hat{k}, \quad \vec{B}(a) = \frac{3}{7}\hat{\imath} - \frac{2}{7}\hat{\jmath} + \frac{6}{7}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = \overset{12\hat{\imath} - \hat{\jmath} + 10\hat{k}}{\cancel{-12\hat{\imath} + \hat{\jmath} - 10\hat{k}}}. \quad \leftarrow \text{mistake!}$$

Find:

(i) $\vec{T}(a)$.

$$\vec{T}(a) = \vec{N}(a) \times \vec{B}(a) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2/7 & -6/7 & -3/7 \\ 3/7 & -2/7 & 6/7 \end{vmatrix}$$

$$= \left(\frac{-36}{7^2} - \frac{6}{7^2}\right)\hat{\imath} - \left(\frac{12}{7^2} + \frac{9}{7^2}\right)\hat{\jmath} + \left(\frac{-4}{7^2} + \frac{18}{7^2}\right)\hat{k} = \frac{-6}{7}\hat{\imath} - \frac{3}{7}\hat{\jmath} + \frac{2}{7}\hat{k}$$

(ii) $\kappa(a)$

Frenet formulae : $\dfrac{d\vec{N}}{ds}(a) = -\kappa\vec{T}(a) + \tau\vec{B}(a)$.

$$\kappa(a) = -\frac{d\vec{N}}{ds}(a) \cdot \vec{T}(a) = -(12, -1, 10) \cdot \left(-\tfrac{6}{7}, -\tfrac{3}{7}, \tfrac{2}{7}\right)$$

$$= \frac{12 \cdot 6 - 3 - 20}{7} = \frac{24 \cdot 2 + 1}{7} = 7$$

(iii) $\tau(a)$

$$\tau(a) = \frac{d\vec{N}}{ds}(a) \cdot \vec{B}(a) = (-12, 1, -10) \cdot \left(\tfrac{3}{7}, -\tfrac{2}{7}, \tfrac{6}{7}\right)$$

$$= \frac{-36 - 2 - 60}{7} = -14.$$

5. (20pts) Let $\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the vector field given by $f(x,y) = y\hat{\imath} - 2\hat{\jmath}$.

(i) Is $\vec{F}$ a gradient field (that is, is $\vec{F}$ conservative)? Why?

$\frac{\partial F_1}{\partial y} = 1 \neq 0 = \frac{\partial F_2}{\partial x}$. So $\vec{F}$ is not conservative

(ii) Is $\vec{F}$ incompressible?

$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 0 = 0$. Yes $\vec{F}$ is incompressible

(iii) Find a flow line that passes through the point (a,b).

Solve

$x'(t) = y(t) \qquad x(0) = a$

$y'(t) = -2 \qquad y(0) = b.$

$y(t) = -2t + b$

$x'(t) = -2t + b$

$x(t) = -t^2 + bt + a$

Sol^n $\quad x(t) = -t^2 + bt + a$

$y(t) = -2t + b.$