FIRST MIDTERM
MATH 18.022, MIT, AUTUMN 10

You have 50 minutes. This test is closed book, closed notes, no calculators.

Name: MODEL ANSWERS

Signature:

Recitation Time:

There are 5 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20pts) (i) Suppose that the four vectors $\vec{t}$, $\vec{u}$, $\vec{v}$ and $\vec{w}$ lie in the same plane Π. Show that

$$(\vec{t} \times \vec{u}) \times (\vec{v} \times \vec{w}) = \vec{0}.$$

$\vec{t} \times \vec{u}$ is orthogonal to both $\vec{t}$ and $\vec{u}$.
So $\vec{t} \times \vec{u}$ is normal to the plane Π.
Similarly $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$,
so $\vec{v} \times \vec{w}$ is normal to the plane Π.
But then $\vec{t} \times \vec{u}$ and $\vec{v} \times \vec{w}$ are parallel
so that $(\vec{t} \times \vec{u}) \times (\vec{v} \times \vec{w}) = \vec{0}$.

(ii) Now suppose that $\vec{t}$, $\vec{u}$, $\vec{v}$ and $\vec{w}$ are four non-zero vectors in $\mathbb{R}^3$, such that

$$(\vec{t} \times \vec{u}) \times (\vec{v} \times \vec{w}) = \vec{0}.$$

Is it true that these four vectors have to lie in the same plane? If true, explain why and if false, give a counterexample.

No, it is not true.
Take $\vec{t} = \vec{u} = \hat{\imath}$, $\vec{v} = \hat{\jmath}$, $\vec{w} = \vec{k}$
Then these vectors don't lie in the same plane, ~~so~~ But $\vec{t} \times \vec{u} = \hat{\imath} \times \hat{\imath} = \vec{0}$
so that $(\vec{t} \times \vec{u}) \times (\vec{v} \times \vec{w}) = \vec{0}$.

2. (20pts) (i) Find a parametric equation for the line l through the two points $P = (1, -1, 2)$ and $Q = (-1, 3, 3)$.

$\overrightarrow{PQ} = (-2, 4, 1)$ If $R = (x, y, z)$ is any pt on the line, then $\overrightarrow{PR} = t\,\overrightarrow{PQ}$, some t.

So $(x-1, y+1, z-2) = t(-2, 4, 1)$

$$(x, y, z) = (1-2t,\ 4t-1,\ t+2)$$

(ii) Find the distance between the line l and the line m given parametrically by $(x, y, z) = (t - 1, 2t + 1, 3 - t)$.

$\vec{v} = (-2, 4, 1)$ $\vec{w} = (1, 2, -1)$

normal $\vec{n}$ to both lines $= \vec{v} \times \vec{w} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} \hat{k}$$

$$= -6\hat{i} - \hat{j} - 8\hat{k}$$

Take $\vec{n} = 6\hat{i} + \hat{j} + 8\hat{k}$

$P = (1, -1, 2)$ $P' = (-1, 1, 3)$ pts on both lines

R, R' closest pts then $\overrightarrow{RR'} = \text{proj}_{\vec{n}}\, \overrightarrow{PP'} = \frac{-2}{101}(6, 1, 8)$

$\overrightarrow{PP'} = (-2, 2, 1)$ $\vec{n} \cdot \overrightarrow{PP'} = -12 + 2 + 8$

$= \cancel{10} - 2$

$d = \frac{2}{100}(101)^{\frac{1}{2}}$

$\|\vec{n}\|^2 = 36 + 1 + 64$

$= 101$

3. (20pts) (i) Find the volume of the parallelepiped spanned by the vectors $\vec{u} = (1, 2, -3)$, $\vec{v} = (1, -2, 1)$ and $\vec{w} = (-1, -2, -1)$.

Signed volume is equal to the scalar triple product $(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}$

$- 3\begin{vmatrix} 1 & -2 \\ -1 & -2 \end{vmatrix}$

$= 4 - 0 + 12 = 16$

Volume $= 16$

(ii) Do the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a right-handed set or a left-handed set?

Sign of scalar triple product is +ve, so we have a right-handed set.

4. (20pts) Let D be the region inside the sphere of radius $2a$ centred at the origin and outside the cylinder of radius a centred around the z-axis.

(i) Describe the region D in cylindrical coordinates.

outside the cylinder of radius a : $r \geq a$.

inside the sphere of radius 2a : $x^2 + y^2 + z^2 \leq a^2$

$r^2 + z^2 \leq a^2$

$r \geq a, \quad r^2 + z^2 \leq a^2$

(ii) Describe the region D in spherical coordinates.

inside the sphere of radius 2a : $\rho \leq 2a$

$r = z \cos\phi$

outside the cylinder of radium a : $z\cos\phi \geq a$

$\rho \leq 2a, \quad z\cos\phi \geq a.$

5. (20pts) Determine whether or not the following limits exist, and if they do exist, then find the limit. Explain your answer.

(i) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$.

No, limit does not exist.

If we approach $(0,0)$ along line $x=0$ we get $\lim_{y\to 0} \dfrac{0}{y^2} = 0$.

If we approach $(0,0)$ along line $y=x$ we get $\lim_{x\to 0} \dfrac{x^2}{2x^2} = \lim_{x\to 0} \dfrac{1}{2} = \dfrac{1}{2} \neq 0$. So limit does not exist.

(ii) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$.

Yes, limit does exist. Use polar coordinates

$xy = r^2 \cos\theta . \sin\theta \quad \sqrt{x^2+y^2} = r$

So $\lim_{(x,y)\to(0,0)} \left| \dfrac{xy}{\sqrt{x^2+y^2}} \right| = \lim_{r\to 0} |r \cos\theta \sin\theta|$

$\leq \lim_{r\to 0} |r| = 0.$