8. **Limits**

**Definition 8.1.** Let \( P \in \mathbb{R}^n \) be a point. The **open ball of radius** \( \epsilon > 0 \) **about** \( P \) is the set

\[
B_\epsilon(P) = \{ Q \in \mathbb{R}^n \mid \| \overrightarrow{PQ} \| < \epsilon \}.
\]

The **closed ball of radius** \( \epsilon > 0 \) **about** \( P \) is the set

\[
\{ Q \in \mathbb{R}^n \mid \| \overrightarrow{PQ} \| \leq \epsilon \}.
\]

**Definition 8.2.** A subset \( A \subset \mathbb{R}^n \) is called **open** if for every \( P \in A \) there is an \( \epsilon > 0 \) such that the open ball of radius \( \epsilon \) about \( P \) is entirely contained in \( A \),

\[
B_\epsilon(P) \subset A.
\]

We say that \( B \) is **closed** if the complement of \( B \) is open.

Put differently, an open set is a union of open balls. Open balls are open and closed balls are closed. \([0, 1)\) is neither open nor closed.

**Definition 8.3.** Let \( B \subset \mathbb{R}^n \). We say that \( P \in \mathbb{R}^n \) is a **limit point** if for every \( \epsilon > 0 \) the intersection

\[
B_\epsilon(P) \cap B \neq \emptyset.
\]

**Example 8.4.** \( 0 \) is a limit point of

\[
\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subset \mathbb{R}.
\]

**Lemma 8.5.** A subset \( B \subset \mathbb{R}^n \) is closed if and only if \( B \) contains all of its limit points.

**Example 8.6.** \( \mathbb{R}^n - \{0\} \) is open. One can see this directly from the definition or from the fact that the complement \( \{0\} \) is closed.

**Definition 8.7.** Let \( A \subset \mathbb{R}^n \) and let \( P \in \mathbb{R}^n \) be a limit point. Suppose that \( f : A \to \mathbb{R}^m \) is a function.

We say that \( f \) **approaches** \( L \) as \( Q \) **approaches** \( P \) and write

\[
\lim_{Q \to P} f(Q) = L,
\]

if for every \( \epsilon > 0 \) we may find \( \delta > 0 \) such that whenever \( Q \in B_\delta(P) \cap A, Q \neq P, f(Q) \in B_\epsilon(L) \). In this case we call \( L \) the **limit**.

It might help to understand the notion of a limit in terms of a game played between two people. Let’s call the first player Larry and the second player Norman. Larry wants to show that \( L \) is the limit of \( f(Q) \) as \( Q \) approaches \( P \) and Norman does not.

So Norman gets to choose \( \epsilon > 0 \). Once Norman has chosen \( \epsilon > 0 \), Larry has to choose \( \delta > 0 \). The smaller that Norman chooses \( \epsilon > 0 \),
the harder Larry has to work (typically he will have to make a choice of $\delta > 0$ very small).

**Proposition 8.8.** Let $f : A \rightarrow \mathbb{R}^m$ and $g : A \rightarrow \mathbb{R}^m$ be two functions. Let $\lambda \in \mathbb{R}$ be a scalar. If $P$ is a limit point of $A$ and

$$\lim_{Q \rightarrow P} f(Q) = L \quad \text{and} \quad \lim_{Q \rightarrow P} g(Q) = M,$$

then

1. $\lim_{Q \rightarrow P} (f + g)(Q) = L + M$, and
2. $\lim_{Q \rightarrow P} (\lambda f)(Q) = \lambda L$.

Now suppose that $m = 1$.

3. $\lim_{Q \rightarrow P} (fg)(Q) = LM$, and
4. if $M \neq 0$, then $\lim_{Q \rightarrow P} (f/g)(Q) = L/M$.

**Proof.** We just prove (1). Suppose that $\epsilon > 0$. As $L$ and $M$ are limits, we may find $\delta_1$ and $\delta_2$ such that, if $\|Q - P\| < \delta_1$ and $Q \in A$, then $\|f(Q) - L\| < \epsilon/2$ and if $\|Q - P\| < \delta_2$ and $Q \in A$, then $\|g(Q) - M\| < \epsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$. If $\|Q - P\| < \delta$ and $Q \in A$, then

$$\|(f + g)(Q) - L - M\| = \|(f(Q) - L) + (g(Q) - M)\|
\leq \|(f(Q) - L)\| + \|(g(Q) - M)\|
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}
= \epsilon,$$

where we applied the triangle inequality to get from the second line to the third line. This is (1). (2-4) have similar proofs.

**Definition 8.9.** Let $A \subset \mathbb{R}^n$ and let $P \in A$. If $f : A \rightarrow \mathbb{R}^m$ is a function, then we say that $f$ is continuous at $P$, if

$$\lim_{Q \rightarrow P} f(Q) = f(P).$$

We say that $f$ is continuous, if it continuous at every point of $A$.

**Theorem 8.10.** If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial function, then $f$ is continuous.

A similar result holds if $f$ is a rational function (a quotient of two polynomials).

**Example 8.11.** $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is continuous.

Sometimes Larry is very lucky:
Example 8.12. Does the limit
\[ \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x - y}, \]
exist? Here the domain of \( f \) is
\[ A = \{ (x,y) \in \mathbb{R}^2 \mid x \neq y \}. \]
Note \((0,0)\) is a limit point of \( A \). Note that if \((x,y)\in A\), then
\[ \frac{x^2 - y^2}{x - y} = x + y, \]
so that
\[ \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(0,0)} x + y = 0. \]
So the limit does exist.

Norman likes the following result:

Proposition 8.13. Let \( A \subset \mathbb{R}^n \) and let \( B \subset \mathbb{R}^m \). Let \( f: A \to B \) and \( g: B \to \mathbb{R}^l \).
Suppose that \( P \) is a limit point of \( A \), \( L \) is a limit point of \( B \) and \( \lim_{Q\to P} f(Q) = L \) and \( \lim_{M\to L} g(M) = E. \)
Then
\[ \lim_{Q\to P} (g \circ f)(Q) = E. \]

Proof. Let \( \epsilon > 0 \). We may find \( \delta > 0 \) such that if \( \|M - L\| < \delta \), and \( M \in B \), then \( \|g(M) - E\| < \epsilon \). Given \( \delta > 0 \) we may find \( \eta > 0 \) such that if \( \|Q - P\| < \eta \) and \( Q \in A \), then \( |f(Q) - L| < \eta \). But then if \( \|Q - P\| < \eta \) and \( Q \in A \), then \( M = f(Q) \in B \) and \( \|M - L\| < \delta \), so that
\[ \|(g \circ f)(Q) - E\| = \|g(f(Q)) - E\| = \|g(M) - E\| < \epsilon. \]

Example 8.14. Does
\[ \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \]
exist? The answer is no.
To show that the answer is no, we suppose that the limit exists.
Suppose we consider restricting to the \( x \)-axis. Let
\[ f: \mathbb{R} \to \mathbb{R}^2, \]
be given by \( t \rightarrow (t,0) \). As \( f \) is continuous, if we compose we must get a function with a limit,

\[
\lim_{t \to 0} \frac{0}{t^2 + 0} = \lim_{t \to 0} 0 = 0.
\]

Now suppose that we restrict to the line \( y = x \). Now consider the function

\[
f : \mathbb{R} \rightarrow \mathbb{R}^2,
\]

be given by \( t \rightarrow (t,t) \). As \( f \) is continuous, if we compose we must get a function with a limit,

\[
\lim_{t \to 0} \frac{t^2}{t^2 + t^2} = \lim_{t \to 0} \frac{1}{2} = \frac{1}{2}.
\]

The problem is that the limit along two different lines is different. So the original limit cannot exist.

**Example 8.15.** Does the limit

\[
\lim_{(x,y) \to (0,0)} \frac{x^3}{x^2 + y^2},
\]

exist? Let us use polar coordinates. Note that

\[
\frac{x^3}{x^2 + y^2} = \frac{r^3 \cos^3 \theta}{r^2} = r \cos^3 \theta.
\]

So we guess the limit is zero.

\[
\lim_{(x,y) \to (0,0)} \left| \frac{x^3}{x^2 + y^2} \right| = \lim_{r \to 0} |r \cos^3 \theta| \\
\leq \lim_{r \to 0} |r| = 0.
\]

**Example 8.16.** Does the limit

\[
\lim_{(x,y,z) \to (0,0,0)} \frac{xyz}{x^2 + y^2 + z^2},
\]

exist? Same trick, but now let us use spherical coordinates.

\[
\lim_{(x,y,z) \to (0,0,0)} \left| \frac{xyz}{x^2 + y^2 + z^2} \right| = \lim_{\rho \to 0} \left| \frac{\rho^3 \cos^2 \phi \sin \phi \cos \theta \sin \theta}{\rho^2} \right| \\
= \lim_{\rho \to 0} \left| \rho \cos^2 \phi \sin \phi \cos \theta \sin \theta \right| \\
\leq \lim_{\rho \to 0} |\rho| = 0.
\]

Sometimes Norman needs to restrict to more complicated curves than just lines:
Example 8.17. Does the limit
\[ \lim_{(x,y) \to (0,0)} \frac{y}{y + x^2}, \]
exist? If we restrict to the line \( t \to (at, bt) \), then we get
\[ \lim_{t \to 0} \frac{bt}{bt + a^2t^2} = \lim_{t \to 0} \frac{b}{b + at} = 1. \]
But if we restrict to the conic \( t \to (t, at^2) \), then we get
\[ \lim_{t \to 0} \frac{at^2}{at^2 + t^2} = \lim_{t \to 0} \frac{a}{1 + a} = \frac{a}{1 + a}, \]
and the limit changes as we vary \( a \), so that the limit does not exist.

Note that if we start with
\[ \frac{y}{y + x^d}, \]
then Norman even needs to use curves of degree \( d \),
\[ t \to (t, at^d). \]