# 8. Limits

**Definition 8.1.** *Let $P \in \mathbb{R}^n$ be a point. The **open ball of radius** $\epsilon > 0$ **about** $P$ is the set*

$$B_\epsilon(P) = \{\, Q \in \mathbb{R}^n \mid \|\overrightarrow{PQ}\| < \epsilon \,\}.$$

*The **closed ball of radius** $\epsilon > 0$ **about** $P$ is the set*

$$\{\, Q \in \mathbb{R}^n \mid \|\overrightarrow{PQ}\| \leq \epsilon \,\}.$$

**Definition 8.2.** *A subset $A \subset \mathbb{R}^n$ is called **open** if for every $P \in A$ there is an $\epsilon > 0$ such that the open ball of radius $\epsilon$ about $P$ is entirely contained in $A$,*

$$B_\epsilon(P) \subset A.$$

*We say that $B$ is **closed** if the complement of $B$ is open.*

Put differently, an open set is a union of open balls. Open balls are open and closed balls are closed. $[0, 1)$ is neither open nor closed.

**Definition 8.3.** *Let $B \subset \mathbb{R}^n$. We say that $P \in \mathbb{R}^n$ is a **limit point** if for every $\epsilon > 0$ the intersection*

$$B_\epsilon(P) \cap B \neq \varnothing.$$

**Example 8.4.** $0$ *is a limit point of*

$$\{\, \frac{1}{n} \mid n \in \mathbb{N} \,\} \subset \mathbb{R}.$$

**Lemma 8.5.** *A subset $B \subset \mathbb{R}^n$ is closed if and only if $B$ contains all of its limit points.*

**Example 8.6.** $\mathbb{R}^n - \{0\}$ *is open. One can see this directly from the definition or from the fact that the complement $\{0\}$ is closed.*

**Definition 8.7.** *Let $A \subset \mathbb{R}^n$ and let $P \in \mathbb{R}^n$ be a limit point. Suppose that $f\colon A \longrightarrow \mathbb{R}^m$ is a function.*

*We say that $f$ approaches $L$ as $Q$ approaches $P$ and write*

$$\lim_{Q \to P} f(Q) = L,$$

*if for every $\epsilon > 0$ we may find $\delta > 0$ such that whenever $Q \in B_\delta(P) \cap A$, $Q \neq P$, $f(Q) \in B_\epsilon(L)$. In this case we call $L$ the **limit**.*

It might help to understand the notion of a limit in terms of a game played between two people. Let's call the first player Larry and the second player Norman. Larry wants to show that $L$ is the limit of $f(Q)$ as $Q$ approaches $P$ and Norman does not.

So Norman gets to choose $\epsilon > 0$. Once Norman has chosen $\epsilon > 0$, Larry has to choose $\delta > 0$. The smaller that Norman chooses $\epsilon > 0$, the harder Larry has to work (typically he will have to make a choice of $\delta > 0$ very small).

**Proposition 8.8.** *Let $f\colon A \longrightarrow \mathbb{R}^m$ and $g\colon A \longrightarrow \mathbb{R}^m$ be two functions. Let $\lambda \in \mathbb{R}$ be a scalar. If $P$ is a limit point of $A$ and*

$$\lim_{Q\to P} f(Q) = L \qquad \text{and} \qquad \lim_{Q\to P} g(Q) = M,$$

*then*

(1) $\lim_{Q\to P}(f+g)(Q) = L + M$, *and*
(2) $\lim_{Q\to P}(\lambda f)(Q) = \lambda L$.

*Now suppose that $m = 1$.*

(3) $\lim_{Q\to P}(fg)(Q) = LM$, *and*
(4) *if $M \neq 0$, then* $\lim_{Q\to P}(f/g)(Q) = L/M$.

*Proof.* We just prove (1). Suppose that $\epsilon > 0$. As $L$ and $M$ are limits, we may find $\delta_1$ and $\delta_2$ such that, if $\|Q - P\| < \delta_1$ and $Q \in A$, then $\|f(Q) - L\| < \epsilon/2$ and if $\|Q - P\| < \delta_2$ and $Q \in A$, then $\|g(Q) - L\| < \epsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$. If $\|Q - P\| < \delta$ and $Q \in A$, then

$$\begin{aligned}
\|(f+g)(Q) - L - M\| &= \|(f(Q) - L) + (g(Q) - M)\| \\
&\leq \|(f(Q) - L)\| + \|(g(Q) - M)\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

where we applied the triangle inequality to get from the second line to the third line. This is (1). (2-4) have similar proofs. □

**Definition 8.9.** *Let $A \subset \mathbb{R}^n$ and let $P \in A$. If $f\colon A \longrightarrow \mathbb{R}^m$ is a function, then we say that $f$ is continuous at $P$, if*

$$\lim_{Q\to P} f(Q) = f(P).$$

*We say that $f$ is continuous, if it continuous at every point of $A$.*

**Theorem 8.10.** *If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a polynomial function, then $f$ is continuous.*

A similar result holds if $f$ is a rational function (a quotient of two polynomials).

**Example 8.11.** *$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is continuous.*

Sometimes Larry is very lucky:

**Example 8.12.** *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y},$$

*exist? Here the domain of $f$ is*

$$A = \{\, (x,y) \in \mathbb{R}^2 \,|\, x \neq y \,\}.$$

*Note $(0,0)$ is a limit point of $A$. Note that if $(x,y) \in A$, then*

$$\frac{x^2-y^2}{x-y} = x+y,$$

*so that*

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} x+y = 0.$$

*So the limit does exist.*

Norman likes the following result:

**Proposition 8.13.** *Let $A \subset \mathbb{R}^n$ and let $B \subset \mathbb{R}^m$. Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow \mathbb{R}^l$.*

*Suppose that $P$ is a limit point of $A$, $L$ is a limit point of $B$ and*

$$\lim_{Q\to P} f(Q) = L \qquad \textit{and} \qquad \lim_{M\to L} g(M) = E.$$

*Then*

$$\lim_{Q\to P} (g\circ f)(Q) = E.$$

*Proof.* Let $\epsilon > 0$. We may find $\delta > 0$ such that if $\|M - L\| < \delta$, and $M \in B$, then $\|g(M) - E\| < \epsilon$. Given $\delta > 0$ we may find $\eta > 0$ such that if $\|Q - P\| < \eta$ and $Q \in A$, then $|f(Q) - L\| < \eta$. But then if $\|Q - P\| < \eta$ and $Q \in A$, then $M = f(Q) \in B$ and $\|M - L\| < \delta$, so that

$$\begin{aligned} \|(g\circ f)(Q) - E\| &= \|g(f(Q)) - E\| \\ &= \|g(M) - E\| \\ &< \epsilon. \end{aligned}$$

∎

**Example 8.14.** *Does*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

*exist? The answer is no.*

*To show that the answer is no, we suppose that the limit exists. Suppose we consider restricting to the $x$-axis. Let*

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

*be given by* $t \longrightarrow (t, 0)$. *As* $f$ *is continuous, if we compose we must get a function with a limit,*

$$\lim_{t\to 0} \frac{0}{t^2+0} = \lim_{t\to 0} 0 = 0.$$

*Now suppose that we restrict to the line* $y = x$. *Now consider the function*

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

*be given by* $t \longrightarrow (t, t)$. *As* $f$ *is continuous, if we compose we must get a function with a limit,*

$$\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}.$$

*The problem is that the limit along two different lines is different. So the original limit cannot exist.*

**Example 8.15.** *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2},$$

*exist? Let us use polar coordinates. Note that*

$$\frac{x^3}{x^2+y^2} = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta.$$

*So we guess the limit is zero.*

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \left|\frac{x^3}{x^2+y^2}\right| &= \lim_{r\to 0} |r\cos^3\theta| \\ &\leq \lim_{r\to 0} |r| = 0.\end{aligned}$$

**Example 8.16.** *Does the limit*

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2},$$

*exist? Same trick, but now let us use spherical coordinates.*

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \left|\frac{xyz}{x^2+y^2+z^2}\right| &= \lim_{\rho\to 0} \left|\frac{\rho^3\cos^2\phi\sin\phi\cos\theta\sin\theta}{\rho^2}\right| \\ &= \lim_{\rho\to 0} |\rho\cos^2\phi\sin\phi\cos\theta\sin\theta| \\ &\leq \lim_{\rho\to 0} |\rho| = 0.\end{aligned}$$

Sometimes Norman needs to restrict to more complicated curves than just lines:

**Example 8.17.** *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{y}{y+x^2},$$

*exist? If we restrict to the line* $t \longrightarrow (at, bt)$*, then we get*

$$\lim_{t\to 0} \frac{bt}{bt+a^2t^2} = \lim_{t\to 0} \frac{b}{b+at} = 1.$$

*But if we restrict to the conic* $t \longrightarrow (t, at^2)$*, then we get*

$$\lim_{t\to 0} \frac{at^2}{at^2+t^2} = \lim_{t\to 0} \frac{a}{1+a} = \frac{a}{1+a},$$

*and the limit changes as we vary* $a$*, so that the limit does not exist.*

Note that if we start with

$$\frac{y}{y+x^d},$$

then Norman even needs to use curves of degree $d$,

$$t \longrightarrow (t, at^d).$$