6. Cylindrical and spherical coordinates

Recall that in the plane one can use polar coordinates rather than Cartesian coordinates. In polar coordinates we specify a point using the distance r from the origin and the angle θ with the x-axis.

In polar coordinates, if a is a constant, then $r = a$ represents a circle of radius a, centred at the origin, and if α is a constant, then $\theta = \alpha$ represents a half ray, starting at the origin, making an angle α.

Suppose that $r = a\theta$, a a constant. This represents a spiral (in fact, the Archimedes spiral), starting at the origin. The smaller a, the 'tighter' the spiral.

By convention, if r is negative, we use this to mean that we point in the opposite direction to the direction given by θ. Also by convention, θ and $\theta + 2\pi$ represent the same point. We may require $r \geq 0$ and $0 \leq \theta < 2\pi$ and if we are not at the origin, this gives us unique polar coordinates.

It is straightforward to convert to and from polar coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

and

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= y/x. \end{aligned}$$

For example, what curve does the equation $r = 2a\cos\theta$ represent? Well if we multiply both sides by r, then we get

$$r^2 = 2ar\cos\theta.$$

So we get

$$x^2 + y^2 = 2ax.$$

Completing the square gives

$$(x-a)^2 + y^2 = a^2.$$

So this is a circle radius a, centred at $(a, 0)$. Polar coordinates can be very useful when we have circles or lines through the origin, or there is a lot of radially symmetry.

Instead of using the vectors $\hat{\imath}$ and $\hat{\jmath}$, in polar coordinates it makes sense to use orthogonal vectors of unit length, that move as the point moves (these are called **moving frames**). At a point P in the plane, with polar coordinates (r, θ), we use the vector $\hat{e}_r$ to denote the vector of unit length pointing in the radial direction:

$$\hat{e}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}.$$

$\hat{e}_r$ points in the direction of increasing r. The vector $\hat{e}_\theta$ is a unit vector pointing in the direction of increasing θ. It is orthogonal to $\hat{e}_r$ and so in fact

$$\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}.$$

We will call a set of unit vectors which are pairwise orthogonal, an **orthonormal** basis if we have two in the plane or three in space.

We want do something similar in space but now there are two choices beyond Cartesian coordinates. The first just takes polar coordinates in the xy-plane and throws in the extra variable z. So a point P is specified by three coordinates, (r, θ, z). r is the distance to the origin, of the projection P' of P down to the xy-plane, θ is the angle $\overrightarrow{OP'}$ makes with the x-axis, so that (r, θ) are just polar coordinates for the point P' in the xy-plane, and z is just the height of P from the xy-plane.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

Note that the locus $r = a$, specifies a cylinder in three space. For this reason we call (r, θ, z) cylindrical coordinates. The locus $\theta = \alpha$, specifies a half-plane which is vertical (if we allow $r < 0$ then we get the full vertical plane). The locus $z = a$ specifies a horizontal plane, parallel to the xy-plane.

The locus $z = ar$ specifies a half cone. At height one, the cone has radius a, so the larger a the more 'open' the cone.

The locus $z = a\theta$ is rather complicated. If we fix the angle, then we get a line of this height and this angle. The resulting surface is called a helicoid, and looks a little bit like a spiral staircase.

Again it is useful to write down an orthonormal coordinate frame. In this case there are three vectors, pointing in the direction of increasing r, increasing θ and increasing z:

$$\begin{aligned} \hat{e}_r &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \\ \hat{e}_\theta &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \\ \hat{e}_z &= \hat{k}. \end{aligned}$$

The third coordinate system in space uses two angles and the distance to the origin, (ρ, θ, ϕ). ρ is the distance to the origin, θ is the angle made by the projection of P down to the xy-plane and ϕ is the angle the radius vector makes with the z-axis. Typically we use coordinates such that $0 \le z \le \infty$, $0 \le \theta < 2\pi$ and $0 \le \phi \le \pi$. To get from spherical coordinates to Cartesian coordinates, we first convert to

cylindrical coordinates,

$$\begin{aligned} r &= \rho \sin\phi \\ \theta &= \theta \\ z &= \rho\cos\phi. \end{aligned}$$

So, in Cartesian coordinates we get

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi. \end{aligned}$$

The locus $z = a$ represents a sphere of radius a, and for this reason we call (ρ, θ, ϕ) cylindrical coordinates. The locus $\phi = a$ represents a cone.

Example 6.1. *Describe the region*

$$x^2 + y^2 + z^2 \leq a^2 \qquad \textit{and} \qquad x^2 + y^2 \geq z^2,$$

in spherical coordinates. The first region is the region inside the sphere of radius,

$$\rho \leq a.$$

The second is the region outside a cone. The surface of the cone is given by $z^2 = x^2 + y^2$. *Now one point on this cone is the point* $(1, 1, 1)$, *so that this a right-angled cone, and the region is given by*

$$\pi/4 \leq \phi \leq 3\pi/4.$$

So we can describe this region by the inequalities

$$\rho \leq a \qquad \textit{and} \qquad \pi/4 \leq \phi \leq 3\pi/4.$$

Finally, let's write down the moving frame given by spherical coordinates, the one corresponding to increasing ρ, increasing θ and increasing ϕ.

$$\begin{aligned} \hat{e}_\rho &= \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \\ &= \sin\phi\cos\theta\hat{\imath} + \sin\phi\sin\theta\hat{\jmath} + \cos\phi\hat{k}. \\ \hat{e}_\theta &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}. \end{aligned}$$

To calculate $\hat{e}_\phi$, we use the fact that it has unit length and it is orthogonal to both $\hat{e}_\rho$ and $\hat{e}_\theta$. We have

$$\begin{aligned}
\hat{e}_\phi &= \pm\hat{e}_\theta \times \hat{e}_\rho \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{vmatrix} \\
&= \cos\phi\cos\theta\hat{\imath} + \sin\theta\cos\phi\hat{\jmath} - (sin^2\theta\sin\phi + \cos^2\theta\sin\phi)\hat{k} \\
&= \cos\phi\cos\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath} - \sin\phi\hat{k}
\end{aligned}$$

Now when ϕ increases, z decreases. So we want the vector with negative z-component, which is exactly the last vector we wrote down.