34. Forms on $\mathbb{R}^n$

Definition 34.1. *A **basic** 1-**form** on $\mathbb{R}^n$ is a formal symbol*

$$dx_1, dx_2, \dots, dx_n.$$

*A **general** 1-**form** on $\mathbb{R}^n$ is any expression of the form*

$$\omega = f_1 \,\mathrm{d}x_1 + f_2 \,\mathrm{d}x_2 + \cdots + f_n \,\mathrm{d}x_n,$$

where $f_1, f_2, \dots, f_n$ are smooth functions.

Note that there are n basic 1-forms. If f is a smooth function, we get a 1-form using the formal rule,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_i.$$

Definition 34.2. *A **basic** 2-**form** on $\mathbb{R}^n$ is any formal symbol*

$$\mathrm{d}x_i \wedge \mathrm{d}x_j,$$

*where $1 \leq i < j \leq n$. A **general** 2-**form** on $\mathbb{R}^n$ is any expression of the form*

$$\omega = \sum_{i<j} f_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j$$

where $f_1, f_2, \dots, f_n$ are smooth functions.

Note that there are

$$\binom{n}{2} = \frac{n(n-1)}{2},$$

basic 2-forms. 2-forms satisfy the following formal rule,

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

Given two 1-forms,

$$\omega = f_1 \,\mathrm{d}x_1 + f_2 \,\mathrm{d}x_2 + \cdots + f_n \,\mathrm{d}x_n \qquad \text{and} \qquad \eta = g_1 \,\mathrm{d}x_1 + g_2 \,\mathrm{d}x_2 + \cdots + g_n \,\mathrm{d}x_n,$$

we can multiply, using linearity and skew-commutativity, to get a 2-form

$$\omega \wedge \eta = \sum_{i<j} (f_i g_j - f_j g_i) \,\mathrm{d}x_i \wedge \mathrm{d}x_j.$$

Notice how, using skew-commutativity, we can write everything in terms of basic 2-forms. Note also that skew-commutativity forces

$$\mathrm{d}x_i \wedge \mathrm{d}x_i = 0.$$

More generally:

Definition 34.3. *A* ***basic*** ***k-form*** *on $\mathbb{R}^n$ is a formal symbol*

$$dx_{i_1} \wedge dx_{i_2} \wedge \dots dx_{i_k},$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ A ***general*** ***k-form*** *on $\mathbb{R}^n$ is any expression of the form*

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f_{i_1 i_2 \dots i_k} \, \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \dots \mathrm{d}x_{i_k},$$

where $f_{i_1 i_2 \dots i_k}$ are smooth functions.

Note that there are

$$\binom{n}{k},$$

basic n-forms.

Suppose we start with a general 1-form in $\mathbb{R}^3$,

$$\omega = f_1 \, \mathrm{d}x + f_2 \, \mathrm{d}y + f_3 \, \mathrm{d}z.$$

We can formally define a 2-form $\mathrm{d}\omega$ as follows:

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(f_1 \, \mathrm{d}x + f_2 \, \mathrm{d}y + f_3 \, \mathrm{d}z) \\
&= \mathrm{d}f_1 \wedge \mathrm{d}x + \mathrm{d}f_2 \wedge \mathrm{d}y + \mathrm{d}f_3 \wedge \mathrm{d}z \\
&= \left(\frac{\partial f_1}{\partial x} \mathrm{d}x + \frac{\partial f_1}{\partial y} \mathrm{d}y + \frac{\partial f_1}{\partial z} \mathrm{d}z \right) \wedge \mathrm{d}x \\
&+ \left(\frac{\partial f_2}{\partial x} \mathrm{d}x + \frac{\partial f_2}{\partial y} \mathrm{d}y + \frac{\partial f_2}{\partial z} \mathrm{d}z \right) \wedge \mathrm{d}y \\
&+ \left(\frac{\partial f_3}{\partial x} \mathrm{d}x + \frac{\partial f_3}{\partial y} \mathrm{d}y + \frac{\partial f_3}{\partial z} \mathrm{d}z \right) \wedge \mathrm{d}z \\
&= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}z + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

Note also that if

$$\omega = f_{12} \, \mathrm{d}x \wedge \mathrm{d}y + f_{13} \, \mathrm{d}x \wedge \mathrm{d}z + f_{23} \, \mathrm{d}y \wedge \mathrm{d}z,$$

then we can formally define a 3-form $\mathrm{d}\omega$ as follows:

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(f_{12} \, \mathrm{d}x \wedge \mathrm{d}y + f_{13} \, \mathrm{d}x \wedge \mathrm{d}z + f_{23} \, \mathrm{d}y \wedge \mathrm{d}z,) \\
&= \left(\frac{\partial f_{12}}{\partial z} - \frac{\partial f_{13}}{\partial y} + \frac{\partial f_{23}}{\partial x} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

The final interesting formal operation on k-forms is:

Definition 34.4. *Let ω be a k-form on $\mathbb{R}^n$. The* ***Hodge star operator*** *assigns an $(n-k)$-form $*\omega$ to ω, which is defined by the rule:*

$$(*\omega) \wedge \omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n.$$

Let's figure out what the Hodge star operator does in $\mathbb{R}^3$. Suppose that

$$\omega = \mathrm{d}x \wedge \mathrm{d}y.$$

Then $\eta = *\omega$ is a 1-form such that

$$\eta \wedge \omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

It follows that $\eta = \mathrm{d}z$. Similarly

$$*(\mathrm{d}x \wedge \mathrm{d}z) = -\mathrm{d}y \qquad \text{and} \qquad * \,(\mathrm{d}y \wedge \mathrm{d}z = \mathrm{d}x).$$

Let

$$\omega = f_1 \,\mathrm{d}x + f_2 \,\mathrm{d}y + f_3 \,\mathrm{d}z.$$

Putting all of this together, we see that

$$\begin{aligned} *\mathrm{d}\omega &= *\mathrm{d}(f_1 \,\mathrm{d}x + f_2 \,\mathrm{d}y + f_3 \,\mathrm{d}z) \\ &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) \mathrm{d}x - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) \mathrm{d}y + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \mathrm{d}z. \end{aligned}$$

Similarly

$$\begin{aligned} *\mathrm{d} * \omega &= *\mathrm{d} * (f_1 \,\mathrm{d}x + f_2 \,\mathrm{d}y + f_3 \,\mathrm{d}z) \\ &= *\mathrm{d}(f_1 \,\mathrm{d}y \wedge \mathrm{d}z - f_2 \,\mathrm{d}x \wedge \mathrm{d}z + f_3 \,\mathrm{d}x \wedge \mathrm{d}y) \\ &= * \left(\frac{\partial f_3}{\partial z} + \frac{\partial f_2}{\partial y} + \frac{\partial f_1}{\partial x}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial x}. \end{aligned}$$

In other words,

$$\mathrm{curl} = *\mathrm{d} \qquad \text{and} \qquad \mathrm{div} = *\mathrm{d} * .$$

Note that this points to a way to unify the various versions of the fundamental theorem of calculus for surfaces and threefolds (namely, the theorems of Green, Stokes and Gauss). One can also use this notation to express Maxwell's equations quite efficiently. But these are both other stories.