Definition 32.1. We say that a vector field
\[ \vec{F} : A \rightarrow \mathbb{R}^m, \]
has compact support if there is a compact subset \( K \subset A \) such that
\[ \vec{F}(\vec{x}) = \vec{0}, \]
for every \( \vec{x} \in A - K \).

If \( S \subset \mathbb{R}^3 \) is a smooth manifold (possibly with boundary) then we will call \( S \) a surface. An orientation is a “continuous” choice of unit normal vector. Not every surface can be oriented. Consider for example the Möbius band, which is obtained by taking a piece of paper and attaching it to itself, except that we add a twist.

Theorem 32.2 (Stokes’ Theorem). Let \( S \subset \mathbb{R}^3 \) be a smooth oriented surface with boundary and let \( \vec{F} : S \rightarrow \mathbb{R}^3 \) be a smooth vector field with compact support.

Then
\[ \int \int_S \text{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}, \]
where \( \partial S \) is oriented compatibly with the orientation on \( S \).

Example 32.3. Let \( S \) be a smooth 2-manifold that looks like a pair of pants. Choose the orientation of \( S \) such that the normal vector is pointing outwards. There are three oriented curves \( C_1, C_2 \) and \( C_3 \) (the two legs and the waist). Suppose that we are given a vector field \( \vec{B} \) with zero curvature. Then (32.2) says that
\[ \int_{C_3} \vec{B} \cdot d\vec{s} + \int_{C_1} \vec{B} \cdot d\vec{s} + \int_{C_2} \vec{B} \cdot d\vec{s} = \int \int_S \text{curl} \vec{B} \cdot d\vec{S} = 0. \]
Here \( C_1' \) and \( C_2' \) denote the curves \( C_1 \) and \( C_2 \) with the opposite orientation. In other words,
\[ \int_{C_3} \vec{B} \cdot d\vec{s} = \int_{C_1} \vec{B} \cdot d\vec{s} + \int_{C_2} \vec{B} \cdot d\vec{s}. \]

Proof of (32.2). We prove this in three steps, in very much the same way as we proved Green’s Theorem.

Step 1: We suppose that \( M = \mathbb{H}^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \), where the plane is the \( xy \)-plane. In this case, we can take \( \hat{n} = \hat{k} \), and this induces the standard orientation of the boundary. Note that
\[ \text{curl} \vec{F} \cdot \hat{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}, \]
and so the result reduces to Green’s Theorem. This completes step 1.

**Step 2:** We suppose that there is a compact subset $K \subset S$ and a parametrisation

$$\vec{g}: \mathbb{H}^2 \cap U \longrightarrow S \cap W,$$

which is compatible with the orientation, such that

1. $\vec{F}(\vec{x}) = \vec{0}$ if $\vec{x} \in S - K$, and
2. $K \subset S \cap W$.

Define a vector field $\vec{G}: \mathbb{H}^2 \longrightarrow \mathbb{R}^2$ by the rule

$$\vec{G}(u, v) = \begin{cases} \vec{F}(\vec{g}(u, v)) \cdot D\vec{g}(u, v) & (u, v) \in U \\ \vec{0} & (u, v) \not\in U. \end{cases}$$

Note that

$$G_1(u, v) = F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u},$$

$$G_2(u, v) = F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v}.$$ 

Using step 1, it is enough to prove:

**Claim 32.4.**

1. $$\iint_S \text{curl} \vec{F} \cdot d\vec{S} = \iint_{\mathbb{H}^2} \left( \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} \right) du \, dv.$$

2. $$\iint_{\partial S} \vec{F} \cdot d\vec{s} = \iint_{\partial \mathbb{H}^2} \vec{G} \cdot d\vec{s}.$$ 

**Proof of 32.4.** Note that

$$\text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} - \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}.$$ 

On the other hand,

$$\frac{\partial \vec{g}}{\partial u} = \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} + \frac{\partial z}{\partial u} \hat{k},$$

$$\frac{\partial \vec{g}}{\partial v} = \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j} + \frac{\partial z}{\partial v} \hat{k}.$$
It follows that
\[
\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}
\end{vmatrix}
= \frac{\partial (y, z)}{\partial (u, v)} \hat{i} - \frac{\partial (x, z)}{\partial (u, v)} \hat{j} + \frac{\partial (x, y)}{\partial (u, v)} \hat{k}.
\]

So,
\[
\text{curl} \vec{F} \cdot \frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \frac{\partial (y, z)}{\partial (u, v)} + \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \frac{\partial (x, z)}{\partial (u, v)} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \frac{\partial (x, y)}{\partial (u, v)}.
\]

On the other hand, if one looks at the proof of the second step of Green’s theorem, we see that
\[
\frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v},
\]
is also equal to the RHS (in fact, what we calculated in the proof of Green’s theorem was the third term of the RHS; by symmetry the other two terms have the same form). This is (1).

For (2), let’s parametrise \(\partial \mathbb{H}^2 \cap U\) by \(\vec{x}(u) = (u, 0)\) and \(\partial S \cap W\) by \(\vec{s}(u) = \vec{g}(\vec{x}(u))\). Then
\[
\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{\partial S \cap W} \vec{F} \cdot d\vec{s}
= \int_a^b \vec{F}(\vec{s}(u)) \cdot \vec{s}'(u) \, du
= \int_a^b \vec{F}(\vec{g}(\vec{x}(u))) \cdot D\vec{g}(\vec{x}(u)) \vec{x}'(u) \, du
= \int_a^b \vec{G}(\vec{x}(u)) \cdot \vec{x}'(u) \, du
= \int_{\partial \mathbb{H}^2 \cap U} \vec{G} \cdot d\vec{s}
= \int_{\partial \mathbb{H}^2} \vec{G} \cdot d\vec{s},
\]
and this is (2).
\[\square\]

This completes step 2.

**Step 3:** We again use partitions of unity. It is straightforward to cover the bounded set \(K\) by finitely many compact subsets \(K_1, K_2, \ldots, K_k\), such that given any smooth vector field which is zero outside \(K_i\), then
the conditions of step 2 hold. By using a partition of unity, we can find smooth functions $\rho_1, \rho_2, \ldots, \rho_k$ such that $\rho_i$ is zero outside $K_i$ and

$$1 = \sum_{i=1}^{k} \rho_i.$$  

Multiplying both sides of this equation by $\vec{F}$, we have

$$\vec{F} = \sum_{i=1}^{k} \vec{F}_i,$$

where $\vec{F}_i = \rho_i \vec{F}$ is a smooth vector field, which is zero outside $K_i$. In this case

$$\iint_S \text{curl} \vec{F} \cdot d\vec{S} = \sum_{i=1}^{k} \iint_S \text{curl} \vec{F}_i \cdot d\vec{S} = \sum_{i=1}^{k} \int_{\partial M} \vec{F}_i \cdot d\vec{s} = \int_{\partial M} \vec{F} \cdot d\vec{s}. \quad \square$$