

30. SURFACE INTEGRALS

Suppose we are given a smooth 2-manifold $M \subset \mathbb{R}^3$. Let

$$\vec{g}: U \longrightarrow M \cap W,$$

be a diffeomorphism, where $U \subset \mathbb{R}^2$, with coordinates s and t .

We can define two tangent vectors, which span the tangent plane to M at $P = \vec{g}(s_0, t_0)$:

$$\begin{aligned}\vec{T}_s(s_0, t_0) &= \frac{\partial \vec{g}}{\partial s}(s_0, t_0) \\ \vec{T}_t(s_0, t_0) &= \frac{\partial \vec{g}}{\partial t}(s_0, t_0).\end{aligned}$$

We get an element of area on M ,

$$dS = \|\vec{T}_s \times \vec{T}_t\| ds dt.$$

Using this we can define the area of $M \cap W$ to be

$$\text{area}(M \cap W) = \iint_{M \cap W} dS = \iint_U \|\vec{T}_s \times \vec{T}_t\| ds dt.$$

Example 30.1. *We can parametrise the torus,*

$$M = \{ (x, y, z) \mid (a - \sqrt{x^2 + y^2})^2 + z^2 = b^2 \},$$

as follows. Let

$$U = (0, 2\pi) \times (0, 2\pi),$$

and

$$W = \mathbb{R}^3 \setminus \{ (x, y, z) \mid x \geq 0 \text{ and } y = 0, \text{ or } x^2 + y^2 \geq a^2 \text{ and } z = 0 \}.$$

Let

$$\vec{g}: U \longrightarrow M \cap W,$$

be the function

$$\vec{g}(s, t) = ((a + b \cos t) \cos s, (a + b \cos t) \sin s, b \sin t).$$

Let's calculate the tangent vectors,

$$\begin{aligned}\vec{T}_s &= \frac{\partial \vec{g}}{\partial s} = (-(a + b \cos t) \sin s, (a + b \cos t) \cos s, 0), \\ \vec{T}_t &= \frac{\partial \vec{g}}{\partial t} = (-b \sin t \cos s, -b \sin t \sin s, b \cos t).\end{aligned}$$

So

$$\begin{aligned}\vec{T}_s \times \vec{T}_t &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(a + b \cos t) \sin s & (a + b \cos t) \cos s & 0 \\ -b \sin t \cos s & -b \sin t \sin s & b \cos t \end{vmatrix} \\ &= (a + b \cos t)b \cos s \cos t \hat{i} + (a + b \cos t)b \sin s \cos t \hat{j} + (a + b \cos t)b \sin t \hat{k}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\vec{T}_s \times \vec{T}_t\| &= (a + b \cos t)b(\cos^2 s \cos^2 t + \sin^2 s \cos^2 t + \sin^2 t)^{1/2} \\ &= (a + b \cos t)b.\end{aligned}$$

As $a \geq b$, note that $(a + b \cos t)b > 0$. Hence

$$\begin{aligned}\text{area}(M) &= \text{area}(M \cap W) \\ &= \iint_{M \cap W} dS \\ &= \iint_U \|\vec{T}_s \times \vec{T}_t\| ds dt \\ &= \int_0^{2\pi} \int_0^{2\pi} (a + b \cos t)b ds dt \\ &= 2\pi b \int_0^{2\pi} (a + b \cos t) dt \\ &= 4\pi^2 ab.\end{aligned}$$

Notice that this is the surface area of a cylinder of radius b and height $2\pi a$, as expected.

Example 30.2. We can parametrise the sphere,

$$M = \{ (x, y, z) \mid x^2 + y^2 + z^2 = a^2 \},$$

as follows. Let

$$U = (0, \pi) \times (0, 2\pi),$$

and

$$W = \mathbb{R}^3 \setminus \{ (x, y, z) \mid x \geq 0 \text{ and } y = 0 \}.$$

Let

$$\vec{g}: U \longrightarrow M \cap W,$$

be the function

$$\vec{g}(\phi, \theta) = (a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi).$$

Let's calculate the tangent vectors,

$$\begin{aligned}\vec{T}_\phi &= \frac{\partial \vec{g}}{\partial \phi} = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi), \\ \vec{T}_\theta &= \frac{\partial \vec{g}}{\partial \theta} = (-a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0).\end{aligned}$$

So

$$\begin{aligned}\vec{T}_\phi \times \vec{T}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= a^2 \sin^2 \phi \cos \theta \hat{i} + a^2 \sin^2 \phi \sin \theta \hat{j} + a^2 \cos \phi \sin \phi \hat{k}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\vec{T}_\phi \times \vec{T}_\theta\| &= a^2 \sin \phi (\sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi)^{1/2} \\ &= a^2 \sin \phi.\end{aligned}$$

As $0 < \phi < \pi$, note that $a^2 \sin \phi > 0$. Hence

$$\begin{aligned}\text{area}(M) &= \text{area}(M \cap W) \\ &= \iint_{M \cap W} dS \\ &= \iint_U \|\vec{T}_\phi \times \vec{T}_\theta\| d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi a^2 \sin \phi d\phi d\theta \\ &= 2a^2 \int_0^{2\pi} dt \\ &= 4\pi a^2.\end{aligned}$$

Notice that this is the surface area of a sphere of radius a .

Let's now suppose that there are two different ways to parametrise the same piece $M \cap W$ of the manifold M :

$$\vec{g}: U \longrightarrow M \cap W \quad \text{and} \quad \vec{h}: V \longrightarrow M \cap W.$$

Let use (u, v) coordinates for $U \subset \mathbb{R}^2$ and (s, t) coordinates for $V \subset \mathbb{R}^2$. Then

$$\vec{f} = (\vec{h})^{-1} \circ \vec{g}: U \longrightarrow V,$$

is a diffeomorphism. Note that $\vec{g} = \vec{h} \circ \vec{f}$. We then have

$$\begin{aligned}\frac{\partial \vec{g}}{\partial u}(u, v) &= \frac{\partial(\vec{h} \circ \vec{f})}{\partial u}(u, v) \\ &= \frac{\partial \vec{h}}{\partial s}(s, t) \frac{\partial s}{\partial u}(u, v) + \frac{\partial \vec{h}}{\partial t}(s, t) \frac{\partial t}{\partial u}(u, v).\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial \vec{g}}{\partial v}(u, v) &= \frac{\partial(\vec{h} \circ \vec{f})}{\partial v}(u, v) \\ &= \frac{\partial \vec{h}}{\partial s}(s, t) \frac{\partial s}{\partial v}(u, v) + \frac{\partial \vec{h}}{\partial t}(s, t) \frac{\partial t}{\partial v}(u, v).\end{aligned}$$

$$\begin{aligned}\frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} &= \left(\frac{\partial \vec{h}}{\partial s} \frac{\partial s}{\partial u} + \frac{\partial \vec{h}}{\partial t} \frac{\partial t}{\partial u} \right) \times \left(\frac{\partial \vec{h}}{\partial s} \frac{\partial s}{\partial v} + \frac{\partial \vec{h}}{\partial t} \frac{\partial t}{\partial v} \right) \\ &= \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t} \left(\frac{\partial s}{\partial u} \frac{\partial t}{\partial v} - \frac{\partial s}{\partial v} \frac{\partial t}{\partial u} \right) \\ &= \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t} \frac{\partial(s, t)}{\partial(u, v)}.\end{aligned}$$

It follows that

$$\left\| \frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} \right\| = \left\| \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t} \right\| \left| \frac{\partial(s, t)}{\partial(u, v)} \right|.$$

Hence

$$\begin{aligned}\iint_U \left\| \frac{\partial \vec{g}}{\partial u} \times \frac{\partial \vec{g}}{\partial v} \right\| du dv &= \iint_U \left\| \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t} \right\| \left| \frac{\partial(s, t)}{\partial(u, v)} \right| du dv \\ &= \iint_V \left\| \frac{\partial \vec{h}}{\partial s} \times \frac{\partial \vec{h}}{\partial t} \right\| ds dt.\end{aligned}$$

Notice that the first term is precisely the integral we use to define the area of $M \cap W$. This formula then says that the area is independent of the choice of parametrisation.