29. Conservative vector fields revisited

Let $U \subset \mathbb{R}^2$ be an open subset. Given a smooth function $f\colon U \longrightarrow \mathbb{R}$ we get a smooth vector field by taking $\vec{F} = \operatorname{grad} f$. Given a smooth vector field $\vec{F}\colon U \longrightarrow \mathbb{R}^2$ we get a function by taking $f = \operatorname{curl} \vec{F}$.

Suppose that $M \subset U$ is a smooth 2-manifold with boundary. If we start with $\vec{F}$, then we have

$$\iint_M \operatorname{curl} \vec{F} \, \mathrm{d}A = \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s}.$$

Suppose we start with f, and let C be a smooth oriented curve. Pick a parametrisation,

$$\vec{x}\colon [a, b] \longrightarrow U,$$

such that $\vec{x}(a) = P$ and $\vec{x}(b) = Q$. Then we have

$$\begin{aligned} \int_C \operatorname{grad} f \cdot \mathrm{d}\vec{s} &= \int_a^b \operatorname{grad} f(\vec{x}(t)) \cdot \vec{x}'(t) \, \mathrm{d}t \\ &= \int_a^b \frac{d}{dt} f(\vec{x}(t)) \, \mathrm{d}t \\ &= f(\vec{x}(b)) - f(\vec{x}(a)) \\ &= f(Q) - f(P). \end{aligned}$$

Definition 29.1. *We say that $X \subset \mathbb{R}^n$ is **star-shaped** with respect to $P \in X$, if given any point $Q \in X$ then the point*

$$P + t\overrightarrow{PQ} \in X,$$

for every $t \in [0, 1]$.

In other words, the line segment connecting P to Q belongs to X.

Theorem 29.2. *Let $U \subset \mathbb{R}^2$ be an open and star-shaped let $\vec{F}\colon U \longrightarrow \mathbb{R}^2$ be a smooth vector field.*

The following are equivalent:

(1) $\operatorname{curl} \vec{F} = 0$.
(2) $\vec{F} = \operatorname{grad} f$.

Proof. (2) implies (1) is easy.

We check (1) implies (2). Suppose that U is star-shaped with respect to $P = (x_0, y_0)$. Parametrise the line L from P to $Q = (x, y)$ as follows

$$P + t\overrightarrow{PQ} = (x_0 + t(x - x_0), y_0 + t(y - y_0)) = P_t, \qquad \text{for} \qquad 0 \leq t \leq 1.$$

Define

$$f(x,y) = \int_L \vec{F} \cdot \mathrm{d}\vec{s}$$
$$= \int_0^1 xF_1(P_t) + yF_2(P_t)\,\mathrm{d}t.$$

Then

$$\frac{\partial f}{\partial x} = \int_0^1 \frac{\partial}{\partial x}\left(xF_1(x_0 + t(x - x_0), y_0 + t(y - y_0)) + yF_2(x_0 + t(x - x_0), y_0 + t(y - y_0))\right)\,\mathrm{d}t$$
$$= \int_0^1 F_1(P_t) + tx\frac{\partial F_1}{\partial x}(P_t) + ty\frac{\partial F_2}{\partial x}(P_t)\,\mathrm{d}t.$$

On the other hand,

$$\frac{\partial}{\partial t}\left(tF_1(x_0 + t(x - x_0), y_0 + t(y - y_0))\right) = F_1(P_t) + tx\frac{\partial F_1}{\partial t}F_1(P_t) + ty\frac{\partial F_2}{\partial y}(P_t).$$

Since $\operatorname{curl}\vec{F} = 0$, we have

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x},$$

and so

$$\frac{\partial f}{\partial x} = \int_0^1 \frac{\partial F_1}{\partial t}(P_t)\,\mathrm{d}t = F_1(x,y).$$

Similarly

$$\frac{\partial f}{\partial y} = F_2(x,y).$$

It follows that $\vec{F} = \operatorname{grad} f$. □

Definition 29.3. *Let $\vec{F}\colon U \longrightarrow \mathbb{R}^2$ be a vector field. Define another vector field by the rule*

$$*F\colon U \longrightarrow \mathbb{R}^2 \qquad \textit{where} \qquad *F = (-F_2, F_1).$$

Theorem 29.4 (Divergence theorem in the plane)**.** *Suppose that $M \subset \mathbb{R}^2$ is a smooth 2-manifold with boundary ∂M.*

If $\vec{F}\colon U \longrightarrow \mathbb{R}^2$ is a smooth vector field, then

$$\iint_M \operatorname{div}\vec{F}\,\mathrm{d}A = \int_{\partial M} \vec{F} \cdot \hat{n}\,\mathrm{d}s,$$

where $\hat{n}$ is the unit normal vector of the smooth oriented curve $C = \partial M$ which points out of M.

Proof. Note that

$$\operatorname{curl}(*\vec{F}) = \operatorname{div}\vec{F},$$

and

$$*F \cdot \mathrm{d}\vec{s} = (\vec{F} \cdot \hat{n})\mathrm{d}s,$$

and so the result follows from Green's theorem applied to $*\vec{F}$. □