28. Manifolds with boundary

Definition 28.1. ***Upper half space*** *is the set*

$$\mathbb{H}^m = \{ (x_1, x_2, \dots, x_m) \mid x_m \geq 0 \} \subset \mathbb{R}^m.$$

The ***boundary*** *of* $\mathbb{H}^m$, *is*

$$\partial\mathbb{H}^m = \{ (x_1, x_2, \dots, x_m) \mid x_m = 0 \} \subset \mathbb{M}^m.$$

Definition 28.2. *A subset* $M \subset \mathbb{R}^k$ *is a* ***smooth*** m***-manifold with boundary*** *if for every* $\vec{a} \in M$ *there is an open subset* $W \subset \mathbb{R}^k$ *and an open subset* $U \subset \mathbb{R}^m$, *and a diffeomorphism*

$$\vec{g} \colon \mathbb{H}^m \cap U \longrightarrow M \cap W.$$

The ***boundary*** *of* M *is the set of points* $\vec{a}$ *which map to a point of the boundary of* $\mathbb{H}^m$.

Example 28.3. *The solid ellipse,*

$$M = \{ (x, y) \in \mathbb{R}^2 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \},$$

is a 2-manifold with boundary.

Let

$$U_1 = \{ (u, v) \mid 0 < u < 2\pi, -1 < v < 1 \}$$

and

$$W_1 = \mathbb{R}^2 - \{ (x, 0) \in \mathbb{R}^2 \mid x \geq 0 \}.$$

Define a function

$$\vec{g}_1 \colon U_1 \longrightarrow W_1,$$

by the rule

$$\vec{g}_1(u, v) = (a(1 - v) \cos u, b(1 - v) \sin u).$$

Similarly, let

$$U_2 = \{ (u, v) \mid -\pi < u < \pi, -1 < v < 1 \}$$

and

$$W_2 = \mathbb{R}^2 - \{ (x, 0) \in \mathbb{R}^2 \mid x \leq 0 \}.$$

Define a function

$$\vec{g}_2 \colon U_2 \longrightarrow W_2,$$

by the rule

$$\vec{g}_2(u, v) = (a(1 - v) \cos u, b(1 - v) \sin u).$$

Finally, let

$$U_3 = \{ (u, v) \mid u^2 + (v - b)^2 < b \}$$

and

$$W_3 = \{\, (x, y) \,|\, x^2 + y^2 < b \,\}.$$

Define a function

$$\vec{g}_3 \colon U_3 \longrightarrow W_3,$$

by the rule

$$\vec{g}_3(u, v) = (u, v - b).$$

Let

$$\vec{F} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

be the function

$$\vec{F}(x, y) = (-y, x).$$

Then

$$\begin{aligned} \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s} &= \int_0^{2\pi} (-b \sin t, a \cos t) \cdot (a \cos t, b \sin t)\,\mathrm{d}t \\ &= \int_0^{2\pi} ab\,\mathrm{d}t \\ &= 2\pi ab. \end{aligned}$$

On the other hand,

$$\operatorname{curl} \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2,$$

and so

$$\iint_M \operatorname{curl} \vec{F}\,\mathrm{d}x\,\mathrm{d}y = 2\pi ab.$$

In fact this is not a coincidence:

Theorem 28.4 (Green's Theorem)**.** *Let $M \subset \mathbb{R}^2$ be a smooth 2-manifold with boundary, and let $\vec{F} \colon M \longrightarrow \mathbb{R}^2$ be a smooth vector field such that*

$$\{\, (x, y) \in M \,|\, \vec{F}(x, y) \neq \vec{0} \,\} \subset \mathbb{R}^2,$$

is a bounded subset.

Then

$$\iint_M \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y = \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s}.$$

Here ∂M is oriented, so that M is on the left as we go around ∂M (in the positive direction).

Proof. In the first step we assume that

$$M = \mathbb{H}^2 = \{ (u, v) \mid v \geq 0 \} \subset \mathbb{R}^2.$$

By assumption we may find a and b such that $\vec{F} = \vec{0}$ outside the box

$$[-a/2, a/2] \times [0, b/2] \subset \mathbb{H}^2.$$

So $\vec{F}(u, v) = \vec{0}$ if $u = \pm a$ or $v = b$. Let's calculate the LHS,

$$\begin{aligned}
\iint_{\mathbb{H}^2} \left(\frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v &= \iint_{\mathbb{H}^2} \frac{\partial F_2}{\partial u} \, \mathrm{d}u \, \mathrm{d}v - \iint_{\mathbb{H}^2} \frac{\partial F_1}{\partial v} \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_0^b \int_{-a}^a \frac{\partial F_2}{\partial u} \, \mathrm{d}u \, \mathrm{d}v - \int_{-a}^a \int_0^b \frac{\partial F_1}{\partial v} \, \mathrm{d}v \, \mathrm{d}u \\
&= \int_0^b F_2(a, v) - F_2(-a, v) \, \mathrm{d}v - \int_{-a}^a F_1(u, b) - F_1(u, 0) \, \mathrm{d}u \\
&= \int_{-a}^a F_1(u, 0) \, \mathrm{d}u.
\end{aligned}$$

Okay, now let's parametrise the boundary of the upper half plane,

$$\vec{x} \colon \mathbb{R} \longrightarrow \partial \mathbb{H}^2,$$

by the rule

$$\vec{x}(u) = (u, 0).$$

Then

$$\vec{x}'(u) = \hat{\imath}.$$

Let's calculate the RHS,

$$\begin{aligned}
\int_{\partial \mathbb{H}^2} \vec{F} \cdot \mathrm{d}\vec{s} &= \int_{-a}^a \vec{F}(\vec{x}(u)) \cdot \vec{x}'(u) \, \mathrm{d}u \\
&= \int_{-a}^a \vec{F}(u, 0) \cdot \hat{\imath} \, \mathrm{d}u \\
&= \int_{-a}^a F_1(u, 0) \, \mathrm{d}u.
\end{aligned}$$

So the result holds if $M = \mathbb{H}^2$. This completes the first step.

In the second step, we suppose that there is a diffeomorphism

$$\vec{g} \colon \mathbb{H}^2 \cap U \longrightarrow M \cap W,$$

such that for some positive real numbers a and b, we have

(1) $[-a, a] \times [0, b] \subset \mathbb{H}^2 \cap U$,
(2) $\vec{F} = \vec{0}$ outside $\vec{g}([-a/2, a/2] \times [0, b])$, and
(3) $\det D\vec{g}(u, v) > 0$ for every $(u, v) \in \mathbb{H}^2 \cap U$.

In this case, parametrise $\partial M \cap W$ as follows; define

$$\vec{s}\colon (-a, a) \longrightarrow \partial M \cap W,$$

by the rule

$$\vec{s}(u) = \vec{g}(\vec{x}(u)) = \vec{g}(u, 0).$$

Note that this is compatible with the orientation, as we are assuming that the Jacobian of g is positive.

$$\begin{aligned} \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s} &= \int_{-a}^{a} \vec{F}(\vec{s}(u)) \cdot \vec{s}'(u)\,\mathrm{d}u \\ &= \int_{-a}^{a} \vec{F}(\vec{g}(\vec{x}(u))) D\vec{g}(\vec{x}(u)) \cdot \vec{x}'(u)\,\mathrm{d}u \\ &= \int_{\partial \mathbb{H}^2} \vec{G} \cdot \mathrm{d}\vec{s}, \end{aligned}$$

where

$$\vec{G}\colon \mathbb{H}^2 \longrightarrow \mathbb{R}^2,$$

is defined by the rule

$$\vec{G}(u, v) = \begin{cases} \vec{F}(\vec{g}(u, v)) D\vec{g}(u, v) & \text{if } (u, v) \in U \\ \vec{0} & \text{otherwise.} \end{cases}$$

Now we compute,

$$\begin{aligned} \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} &= \frac{\partial}{\partial u}\left(F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u}\right) \\ &= \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial v} + F_1 \frac{\partial^2 x}{\partial u \partial v} \\ &+ \left(\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_2}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial v} + F_2 \frac{\partial^2 y}{\partial u \partial v} \\ &- \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial x}{\partial u} - F_1 \frac{\partial^2 x}{\partial v \partial u} \\ &- \left(\frac{\partial F_2}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_2}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial y}{\partial u} - F_2 \frac{\partial^2 x}{\partial v \partial u} \\ &= \frac{\partial F_2}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial x}\frac{\partial y}{\partial u}\right) - \frac{\partial F_1}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial x}\frac{\partial y}{\partial u}\right) \\ &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\frac{\partial(x, y)}{\partial(u, v)}. \end{aligned}$$

Using this, we get

$$\begin{aligned}\iint_{\mathbb{H}^2}\left(\frac{\partial G_2}{\partial u}-\frac{\partial G_1}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v &= \iint_{\mathbb{H}^2\cap U}\left(\frac{\partial G_2}{\partial u}-\frac{\partial G_1}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v\\ &= \iint_{\mathbb{H}^2\cap U}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)}\,\mathrm{d}u\,\mathrm{d}v\\ &= \iint_{M\cap W}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\\ &= \iint_{M}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Putting all of this together, we have

$$\begin{aligned}\iint_{M}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y &= \iint_{\mathbb{H}^2}\left(\frac{\partial G_2}{\partial u}-\frac{\partial G_1}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v\\ &= \int_{\partial\mathbb{H}^2}\vec{G}\cdot\mathrm{d}\vec{s}\\ &= \int_{\partial M}\vec{F}\cdot\mathrm{d}\vec{s}.\end{aligned}$$

This completes step 2.

We now turn to the third and final step. To complete the proof, we need to invoke the existence of partitions of unity. Starting with $\vec{F}$, I claim that there are vector finitely many fields $F_1, F_2, \ldots, F_k$, each of which satisfy the hypotheses of step 2, such that

$$\vec{F}=\vec{F}_1+\vec{F}_2+\cdots+\vec{F}_k=\sum_{i=1}^{k}\vec{F}_i.$$

Indeed, start with a partition of unity,

$$1=\sum_{i=1}^{m}\rho_i,$$

multiply both sides by $\vec{F}$, to get

$$\vec{F}=\sum i=1^m\rho_i\vec{F}=\sum_{i=1}^{m}\vec{F}_i.$$

Granted this, Green's Theorem follows very easily,

$$\iint_M \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{k} \iint_M \left(\frac{\partial F_{i,2}}{\partial x} - \frac{\partial F_{i,1}}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

$$= \sum_{i=1}^{k} \int_{\partial M} \vec{F}_i \cdot \mathrm{d}\vec{s}.$$

$$= \int_{\partial M} \vec{F} \cdot \mathrm{d}\vec{s}. \qquad \square$$

Lemma 28.5. *Let $K \subset \mathbb{R}^n$. Suppose that K is containedin the union of closed balls $B_1, B_2, \dots, B_m$, such that any point of K belongs to the interior of at least one of $B_1, B_2, \dots, B_m$.*

Then we may find smooth functions $\rho_1, \rho_2, \dots, \rho_m$ such that ρ_i is zero outside B_i and

$$1 = \sum_{i=1}^{m} \rho_i.$$

Proof. We prove the case $n = 2$. The general case is similar, only notationally more involved. First observe that it is enough to find smooth functions $\sigma_1, \sigma_2, \dots, \sigma_m$, such that σ_i is zero outside B_i and such that

$$\sigma = \sum_{i=1}^{m} \sigma_i,$$

does not vanish at any point of K. Indeed, if we let

$$\rho_i = \frac{\sigma_i}{\sigma},$$

then ρ_i is smooth, it vanishes outside B_i and dividing both sides of the equation above by σ, we have

$$1 = \sum_{i=1}^{m} \rho_i.$$

In fact it suffices to find functions $\sigma_1, \sigma_2, \dots, \sigma_m$, such that σ_i vanishes outside B_i and which is non-zero on the interior of B_i (replacing σ_i by σ_i^2, so that σ_i^2 is positive on the interior of B_i, we get rid of the annoying possibility that the sum is zero because of cancelling). It is enough to do this for one solid circle B_i and we might as well assume that $B = B_m = B_1$ is the solid unit circle. Using polar coordinates, we want a function of one variable r which is zero outside $[0, 1]$ and which is non-zero on $(0, 1)$, so we are now down to a one variable question.

At this point we realise we want a smooth function,

$$f \colon \mathbb{R} \longrightarrow \mathbb{R},$$

all of whose derivatives are zero at 0 and yet the function f is not the zero function. Such a function is given by

$$f(x) = e^{-1/x^2}. \qquad \square$$