27. Line integrals

Let I be an open interval and let

$$\vec{x}\colon I \longrightarrow \mathbb{R}^n,$$

be a parametrised differentiable curve. If $[a, b] \subset I$ then let $C = \vec{x}([a, b])$ be the image of $[a, b]$ and let $f\colon C \longrightarrow \mathbb{R}$ be a function.

Definition 27.1. *The **line integral** of f along C is*

$$\int_C f \, \mathrm{d}s = \int_a^b f(\vec{x}(u))\|\vec{x}'(u)\| \, \mathrm{d}u.$$

Let $u\colon J \longrightarrow I$ be a diffeomorphism between two open intervals. Suppose that u is C^1.

Definition 27.2. *We say that u is **orientation-preserving** if $u'(t) > 0$ for every $t \in J$.*

*We say that u is **orientation-reversing** if $u'(t) < 0$ for every $t \in J$.*

Notice that u is always either orientation-preserving or orientation-reversing (this is a consequence of the intermediate value theorem, applied to the continuous function $u'(t)$).

Define a function

$$\vec{y}\colon J \longrightarrow \mathbb{R}^n,$$

by composition,

$$\vec{y}(t) = \vec{x}(u(t)),$$

so that $\vec{y} = \vec{x} \circ u$.

Now suppose that $u([c, d]) = [a, b]$. Then $C = \vec{y}([c, d])$, so that $\vec{y}$ gives another parametrisation of C.

Lemma 27.3.

$$\int_a^b f(\vec{x}(u))\|\vec{x}'(u)\| \, \mathrm{d}u = \int_c^d f(\vec{y}(t))\|\vec{y}'(t)\| \, \mathrm{d}t.$$

Proof. We deal with the case that u is orientation-reversing. The case that u is orientation-preserving is similar and easier.

As u is orientation-reversing, we have $u(c) = b$ and $u(d) = a$ and so,

$$\begin{aligned}
\int_c^d f(\vec{y}(t))\|\vec{y}'(t)\| \,\mathrm{d}t &= \int_c^d f(\vec{x}(u(t)))\|u(t)\vec{x}'(u(t))\| \,\mathrm{d}t \\
&= -\int_c^d f(\vec{x}(u(t)))\|\vec{x}'(u(t))\|u'(t) \,\mathrm{d}t \\
&= -\int_b^a f(\vec{x}(u))\|\vec{x}'(u)\| \,\mathrm{d}u \\
&= \int_a^b f(\vec{x}(u))\|\vec{x}'(u)\| \,\mathrm{d}u.
\end{aligned}$$

□

Now suppose that we have a vector field on C,

$$\vec{F} \colon C \longrightarrow \mathbb{R}^n.$$

Definition 27.4. *The* ***line integral*** *of $\vec{F}$ along C is*

$$\int_C \vec{F} \cdot \mathrm{d}\vec{s} = \int_a^b \vec{F}(\vec{x}(u)) \cdot \vec{x}'(u) \,\mathrm{d}u.$$

Note that now the orientation is very important:

Lemma 27.5.

$$\int_a^b \vec{F}(\vec{x}(u)) \cdot \vec{x}'(u) \,\mathrm{d}u = \begin{cases} \int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t) \,\mathrm{d}t & u'(t) > 0 \\ -\int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t) \,\mathrm{d}t & u'(t) < 0 \end{cases}$$

Proof. We deal with the case that u is orientation-reversing. The case that u is orientation-preserving is similar and easier.

As u is orientation-reversing, we have $u(c) = b$ and $u(d) = a$ and so,

$$\begin{aligned}
\int_c^d \vec{F}(\vec{y}(t)) \cdot \vec{y}'(t) \,\mathrm{d}t &= \int_c^d \vec{F}(\vec{x}(u(t))) \cdot \vec{x}'(u(t))u'(t) \,\mathrm{d}t \\
&= \int_b^a \vec{F}(\vec{x}(u)) \cdot \vec{x}'(u) \,\mathrm{d}u \\
&= -\int_a^b \vec{F}(\vec{x}(u)) \cdot \vec{x}'(u) \,\mathrm{d}u.
\end{aligned}$$

□

Example 27.6. *If C is a piece of wire and $f(\vec{x})$ is the mass density at $\vec{x} \in C$, then the line integral*

$$\int_C f \,\mathrm{d}s,$$

is the total mass of the curve. Clearly this is always positive, whichever way you parametrise the curve.

Example 27.7. *If C is an oriented path and $\vec{F}(\vec{x})$ is a force field, then the line integral*

$$\int_C \vec{F}\cdot \mathrm{d}\vec{s},$$

is the work done when moving along C. If we reverse the orientation, then the sign flips. For example, imagine C is a spiral staircase and $\vec{F}$ is the force due to gravity. Going up the staircase costs energy and going down we gain energy.

Definition 27.8. *Let $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be two open subsets.*

We say that

$$f\colon U \longrightarrow V,$$

*is **smooth** if all higher order partials*

$$\frac{\partial^n f}{\partial x_{i_1}\dots\partial x_{i_n}}(x_1, x_2, \dots, x_k),$$

exist and are continuous.

Definition 27.9. *Now suppose that $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ are any two subsets. We say that a function*

$$\vec{f}\colon X \longrightarrow Y,$$

*is **smooth**, if given any point $\vec{a} \in X$ we may find $\vec{a} \in U \subset \mathbb{R}^k$ open, and a smooth function*

$$\vec{F}\colon U \longrightarrow \mathbb{R}^l,$$

such that $\vec{f}(\vec{x}) = \vec{F}(\vec{x})$, where $\vec{x} \in X \cap U$ (equivalently $\vec{f}|_{X\cap U} = \vec{F}|_{X\cap U}$), and we put

$$D\vec{f}(\vec{x}) = D\vec{F}(\vec{x}).$$

*We say that $\vec{f}$ is a (smooth) **diffeomorphism** if $\vec{f}$ is bijective and both $\vec{f}$ and $\vec{f}^{-1}$ are smooth.*

Notice that in the definition of a diffeomorphism we are now requiring more than we did (before we just required that $\vec{f}$ and $\vec{f}^{-1}$ were differentiable).

Remark 27.10. *Note that if X is not very "big" then $Df(\vec{x})$ might not be unique. For example, if $X = \{\vec{x}\}$ is a single point, then there are very many different ways to extend $\vec{f}$ to a function $\vec{F}$ in an open neighbourhood of $\vec{x}$. In the examples we consider in this class, this will not be an issue (namely, manifolds with boundary).*

Example 27.11. *The map*

$$\vec{x}\colon [a,b] \longrightarrow \mathbb{R}^n,$$

is smooth if and only if there is a constant $\epsilon > 0$ *and a smooth function*

$$\vec{y}\colon (a-\epsilon, b+\epsilon) \longrightarrow \mathbb{R}^n,$$

whose restriction to $[a,b]$ *is the function* $\vec{x}$*,*

$$\vec{y}(t) = \vec{x}(t) \qquad \text{for all} \qquad t \in [a,b].$$

Lemma 27.12. *If*

$$\vec{x}\colon [a,b] \longrightarrow \mathbb{R}^n,$$

is injective for all $t \in [a,b]$*, then*

$$\vec{x}\colon [a,b] \longrightarrow C = \vec{x}([a,b]),$$

is a diffeomorphism.