26. Change of coordinates: II

Example 26.1. *Let D be the region bounded by the cardiod,*

$$r = 1 - \cos\theta.$$

If we multiply both sides by r, take $r\cos\theta$ over the other side, then we get

$$(x^2 + y^2 + x)^2 = x^2 + y^2.$$

We have

$$
\begin{aligned}
\text{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left(\int_0^{1-\cos\theta} r\,\mathrm{d}r \right) \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left[\frac{r^2}{2} \right]_0^{1-\cos\theta} \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{(1-\cos\theta)^2}{2}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \cos\theta + \frac{\cos^2\theta}{2}\,\mathrm{d}\theta \\
&= \left[\frac{\theta}{2} - \sin\theta \right]_{-\pi}^{\pi} \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{aligned}
$$

In $\mathbb{R}^3$, we can either use cylindrical or spherical coordinates, instead of Cartesian coordinates.

Let's first do the case of cylindrical coordinates. Recall that

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z.
\end{aligned}
$$

So the Jacobian is given by

$$
\frac{\partial(x,y,z)}{\partial(r,\theta,z)}(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.
$$

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So,
$$\iiint_W f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_{W^*} f(r, \theta, z)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z.$$

Example 26.2. *Consider a cone of height b and base radius a. Put the vertex of the cone at the point $(0, 0, b)$, so that the base of the cone is the circle of radius a, centred at the origin, in the xy-plane. Note that at height z, we have a circle of radius*
$$a\left(1 - \frac{z}{b}\right).$$

$$\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\
&= \iiint_{W^*} r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z \\
&= \int_0^b \left(\int_0^{2\pi} \left(\int_0^{a(1-z/b)} r\, \mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}z \\
&= \frac{1}{2} \int_0^b \left(\int_0^{2\pi} [r^2]_0^{a(1-z/b)}\, \mathrm{d}\theta \right) \mathrm{d}z \\
&= \frac{1}{2} \int_0^b \left(\int_0^{2\pi} a^2 \left(1 - \frac{z}{b}\right)^2 \mathrm{d}\theta \right) \mathrm{d}z \\
&= \pi a^2 \int_{-a}^a \left(1 - \frac{z}{b}\right)^2 \mathrm{d}z \\
&= -\pi a^2 b \int_1^0 u^2\, \mathrm{d}u \\
&= \pi a^2 b \int_0^1 u^2\, \mathrm{d}u \\
&= \frac{\pi a^2 b}{3}.
\end{aligned}$$

Example 26.3. *Consider a ball of radius a. Put the centre of the ball at the point $(0, 0, 0)$. Note that*
$$x^2 + y^2 + z^2 = a^2,$$

translates to the equation
$$r^2 + z^2 = a^2,$$

so that
$$r = \sqrt{a^2 - z^2}.$$

$$\text{vol}(W) = \iiint_W \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$= \iint_{W^*} r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$$

$$= \int_{-a}^{a} \left(\int_0^{2\pi} \left(\int_0^{\sqrt{a^2-z^2}} r \, \mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}z$$

$$= \frac{1}{2} \int_{-a}^{a} \left(\int_0^{2\pi} [r^2]_0^{\sqrt{a^2-z^2}} \, \mathrm{d}\theta \right) \mathrm{d}z$$

$$= \frac{1}{2} \int_{-a}^{a} \left(\int_0^{2\pi} a^2 - z^2 \, \mathrm{d}\theta \right) \mathrm{d}z$$

$$= \pi \int_{-a}^{a} a^2 - z^2 \, \mathrm{d}z$$

$$= \pi \left[a^2 z - \frac{z^3}{3} \right]_{-a}^{a}$$

$$= \frac{4\pi a^3}{3}.$$

Now consider using spherical coordinates. Recall that

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi.$$

So

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}(\rho,\phi,\theta) = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$
$$= \rho^2 \cos^2\phi \sin\phi + \rho^2 \sin^3\phi = \rho^2 \sin\phi.$$

Notice that this is greater than zero, if $0 < \phi < \pi$. So,

$$\iiint_W f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iiint_{W^*} f(\rho,\phi,\theta)\rho^2 \sin\phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z.$$

Example 26.4. *Consider a ball of radius a. Put the centre of the ball at the point* $(0,0,0)$.

$$
\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_{W^*} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^a \rho^2 \sin\phi\,\mathrm{d}\rho \right) \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\phi \left[\frac{\rho^3}{3} \right]_0^a \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\phi \frac{a^3}{3}\,\mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} \left[-\cos\phi \right]_0^{\pi} \mathrm{d}\theta \\
&= \frac{2a^3}{3} \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{4\pi a^3}{3}.
\end{aligned}
$$