25. Change of coordinates: I

Definition 25.1. *A function $f\colon U \longrightarrow V$ between two open subsets of $\mathbb{R}^n$ is called a **diffeomorphism** if:*

1. *f is a bijection,*
2. *f is differentiable, and*
3. *f^{-1} is differentiable.*

Almost be definition of the inverse function, $f \circ f^{-1}\colon V \longrightarrow V$ and $f^{-1} \circ f\colon U \longrightarrow U$ are both the identity function, so that

$$(f \circ f^{-1})(\vec{y}) = \vec{y} \qquad \text{and} \qquad (f^{-1} \circ f)(\vec{x}) = \vec{x}.$$

It follows that

$$Df(\vec{x})Df^{-1}(\vec{y}) = I_n \qquad \text{and} \qquad Df^{-1}(\vec{y})Df(\vec{x}) = I_n,$$

by the chain rule. Taking determinants, we see that

$$\det(Df)\det(Df^{-1}) = \det I_n = 1.$$

Therefore,

$$\det(Df^{-1}) = (\det(Df))^{-1}.$$

It follows that

$$\det(Df) \neq 0.$$

Theorem 25.2 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \longrightarrow \mathbb{R}$ be a function.*

Suppose that

1. *f is injective,*
2. *f is $\mathcal{C}^1$, and*
3. *$Df(\vec{x}) \neq 0$ for all $\vec{x} \in U$.*

Then $V = f(U) \subset \mathbb{R}^n$ is open and the induced map $f\colon U \longrightarrow V$ is a diffeomorphism.

Example 25.3. *Let $f(r, \theta) = (r\cos\theta, r\sin\theta)$. Then*

$$Df(r, \theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix},$$

so that

$$\det Df(r, \theta) = r.$$

It follows that f defines a diffeomorphism $f\colon U \longrightarrow V$ between

$$U = (0, \infty) \times (0, 2\pi) \qquad \textit{and} \qquad V = \mathbb{R}^2 \setminus \{\, (x, y) \in \mathbb{R}^2 \mid y = 0, x \geq 0 \,\}.$$

Theorem 25.4. *Let $g\colon U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$,*

$$g(u,v) = (x(u,v), y(u,v)).$$

Let $D^* \subset U$ be a region and let $D = f(D^*) \subset V$. Let $f\colon D \longrightarrow \mathbb{R}$ be a function.

Then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D^*} f(x(u,v), y(u,v))|\det Dg(u,v)|\,\mathrm{d}u\,\mathrm{d}v.$$

It is convenient to use the following notation:

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \det Dg(u,v).$$

The LHS is called the **Jacobian**. Note that

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \left(\frac{\partial(u,v)}{\partial(x,y)}(x,y)\right)^{-1}.$$

Example 25.5. *There is no simple expression for the integral of e^{-x^2}. However it is possible to compute the following integral*

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

(In what follows, we will ignore issues relating to the fact that the integrals are improper; in practice all integrals converge). Instead of

computing I, we compute I^2,

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right)\\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\right)\mathrm{d}y\\
&= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{\mathbb{R}^2} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_0^{\infty}\left(\int_0^{2\pi} re^{-r^2}\,\mathrm{d}\theta\right)\mathrm{d}r\\
&= \int_0^{\infty} re^{-r^2}\left(\int_0^{2\pi}\mathrm{d}\theta\right)\mathrm{d}r\\
&= 2\pi\int_0^{\infty} re^{-r^2}\mathrm{d}r\\
&= 2\pi\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty}\\
&= \pi.
\end{aligned}$$

So $I = \sqrt{\pi}$.

Example 25.6. *Find the area of the region D bounded by the four curves*

$$xy = 1, \qquad xy = 3, \qquad y = x^3, \qquad \text{and} \qquad y = 2x^3.$$

Define two new variables,

$$u = \frac{x^3}{y} \qquad \text{and} \qquad v = xy.$$

Then D is a rectangle in uv-coordinates,

$$D^* = [1/2, 1] \times [1, 3]$$

Now for the Jacobian we have

$$\frac{\partial(u,v)}{\partial(x,y)}(x,y) = \begin{vmatrix} \frac{3x^2}{y} & -\frac{x^3}{y^2} \\ y & x \end{vmatrix} = \frac{4x^3}{y} = 4u.$$

It follows that

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \frac{1}{4u}.$$

This is nowhere zero. In fact note that we can solve for x and y explicitly in terms of u and v.

$$uv = x^4 \qquad \text{and} \qquad y = \frac{x}{v}.$$

So

$$x = (uv)^{1/4} \qquad \text{and} \qquad y = u^{-1/4}v^{3/4}.$$

Therefore

$$\begin{aligned}
\operatorname{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} \frac{1}{4u}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 \left(\int_{1/2}^1 \frac{1}{u}\,\mathrm{d}u\right)\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 [\ln u]_{1/2}^1\,\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 \ln 2\,\mathrm{d}v \\
&= \frac{1}{2}\ln 2.
\end{aligned}$$

Theorem 25.7. *Let $g\colon U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^3$,*

$$g(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

Let $W^* \subset U$ be a region and let $W = f(W^*) \subset V$. Let $f\colon W \longrightarrow \mathbb{R}$ be a function.

Then

$$\iiint_W f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w))|\det Dg(u, v, w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

As before, it is convenient to introduce more notation:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}(u, v, w) = \det Dg(u, v, w).$$