25. Change of coordinates: I

Definition 25.1. A function $f : U \rightarrow V$ between two open subsets of $\mathbb{R}^n$ is called a **diffeomorphism** if:

1. $f$ is a bijection,
2. $f$ is differentiable, and
3. $f^{-1}$ is differentiable.

Almost by definition of the inverse function, $f \circ f^{-1} : V \rightarrow V$ and $f^{-1} \circ f : U \rightarrow U$ are both the identity function, so that

$$(f \circ f^{-1})(\vec{y}) = \vec{y} \quad \text{and} \quad (f^{-1} \circ f)(\vec{x}) = \vec{x}.$$ 

It follows that

$$Df(\vec{x})Df^{-1}(\vec{y}) = I_n \quad \text{and} \quad Df^{-1}(\vec{y})Df(\vec{x}) = I_n,$$

by the chain rule. Taking determinants, we see that

$$\det(Df)\det(Df^{-1}) = \det I_n = 1.$$ 

Therefore,

$$\det(Df) = (\det(Df))^{-1}.$$ 

It follows that

$$\det(Df) \neq 0.$$ 

**Theorem 25.2** (Inverse function theorem). Let $U \subset \mathbb{R}^n$ be an open subset and let $f : U \rightarrow \mathbb{R}$ be a function.

Suppose that

1. $f$ is injective,
2. $f$ is $C^1$, and
3. $Df(\vec{x}) \neq 0$ for all $\vec{x} \in U$.

Then $V = f(U) \subset \mathbb{R}^n$ is open and the induced map $f : U \rightarrow V$ is a diffeomorphism.

**Example 25.3.** Let $f(r, \theta) = (r \cos \theta, r \sin \theta)$. Then

$$Df(r, \theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix},$$

so that

$$\det Df(r, \theta) = r.$$ 

It follows that $f$ defines a diffeomorphism $f : U \rightarrow V$ between

$U = (0, \infty) \times (0, 2\pi)$ \quad and \quad $V = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \mid y = 0, x \geq 0 \}$.  

Theorem 25.4. Let \( g: U \longrightarrow V \) be a diffeomorphism between open subsets of \( \mathbb{R}^2 \),

\[
g(u, v) = (x(u, v), y(u, v)).
\]

Let \( D^* \subset U \) be a region and let \( D = f(D^*) \subset V \). Let \( f: D \longrightarrow \mathbb{R} \) be a function.

Then

\[
\iint_D f(x, y) \, dx \, dy = \iint_{D^*} f(x(u, v), y(u, v)) |\det Dg(u, v)| \, du \, dv.
\]

It is convenient to use the following notation:

\[
\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \det Dg(u, v).
\]

The LHS is called the Jacobian. Note that

\[
\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \left( \frac{\partial(u, v)}{\partial(x, y)}(x, y) \right)^{-1}.
\]

Example 25.5. There is no simple expression for the integral of \( e^{-x^2} \).

However it is possible to compute the following integral

\[
I = \int_{-\infty}^{\infty} e^{-x^2} \, dx.
\]

(In what follows, we will ignore issues relating to the fact that the integrals are improper; in practice all integrals converge). Instead of
computing $I$, we compute $I^2$,

$$I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right)$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx \right) \, dy$$

$$= \int_{\mathbb{R}^2} e^{-x^2-y^2} \, dx \, dy$$

$$= \int_{\mathbb{R}^2} re^{-r^2} \, dr \, d\theta$$

$$= \int_0^\infty \left( \int_0^{2\pi} re^{-r^2} \, d\theta \right) \, dr$$

$$= \int_0^\infty re^{-r^2} \left( \int_0^{2\pi} d\theta \right) \, dr$$

$$= 2\pi \int_0^\infty re^{-r^2} \, dr$$

$$= 2\pi \left[ -\frac{e^{-r^2}}{2} \right]_0^\infty$$

$$= \pi.$$

So $I = \sqrt{\pi}$.

**Example 25.6.** Find the area of the region $D$ bounded by the four curves

$$xy = 1, \quad xy = 3, \quad y = x^3, \quad \text{and} \quad y = 2x^3.$$

Define two new variables,

$$u = \frac{x^3}{y} \quad \text{and} \quad v = xy.$$

Then $D$ is a rectangle in $uv$-coordinates,

$$D^* = [1/2, 1] \times [1, 3]$$

Now for the Jacobian we have

$$\frac{\partial(u, v)}{\partial(x, y)}(x, y) = \begin{vmatrix} 3x^2/v & -x^3/y^2 \\ y & x \end{vmatrix} = \frac{4x^4}{y} = 4u.$$

It follows that

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \frac{1}{4u}.$$
This is nowhere zero. In fact note that we can solve for $x$ and $y$ explicitly in terms of $u$ and $v$.

\[ uv = x^4 \quad \text{and} \quad y = \frac{x}{v}. \]

So

\[ x = (uv)^{1/4} \quad \text{and} \quad y = u^{-1/4}v^{3/4}. \]

Therefore

\[
\text{area}(D) = \int\int_D dx \, dy \\
= \int\int_{D^*} \frac{1}{4u} \, du \, dv \\
= \frac{1}{4} \int_1^3 \left( \int_{1/2}^1 \frac{1}{u} \, du \right) \, dv \\
= \frac{1}{4} \int_1^3 \ln u|_{1/2}^1 \, dv \\
= \frac{1}{4} \int_1^3 \ln 2 \, dv \\
= \frac{1}{2} \ln 2.
\]

**Theorem 25.7.** Let $g: U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^3$,

\[ g(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)). \]

Let $W^* \subset U$ be a region and let $W = f(W^*) \subset V$. Let $f: W \longrightarrow \mathbb{R}$ be a function.

Then

\[
\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \, | \det Dg(u, v, w) | \, du \, dv \, dw.
\]

As before, it is convenient to introduce more notation:

\[ \frac{\partial(x, y, z)}{\partial(u, v, w)}(u, v, w) = \det Dg(u, v, w). \]