24. Triple integrals

**Definition 24.1.** Let \( B = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3 \) be a box in space. A partition \( \mathcal{P} \) of \( R \) is a triple of sequences:

\[
\begin{align*}
& a = x_0 < x_1 < \cdots < x_n = b \\
& c = y_0 < y_1 < \cdots < y_n = d \\
& e = z_0 < z_1 < \cdots < z_n = f.
\end{align*}
\]

The mesh of \( \mathcal{P} \) is

\[
m(\mathcal{P}) = \max \{ x_i - x_{i-1}, y_i - y_{i-1}, z_i - z_{i-1} | 1 \leq i \leq k \}.
\]

Now suppose we are given a function

\[
f : B \rightarrow \mathbb{R}
\]

Pick

\[
\vec{c}_{ijk} \in B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{i-1}, z_i].
\]

**Definition 24.2.** The sum

\[
S = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(\vec{c}_{ijk}) (x_i - x_{i-1})(y_j - y_{j-1})(z_i - z_{i-1}),
\]

is called a **Riemann sum**.

**Definition 24.3.** The function \( f : B \rightarrow \mathbb{R} \) is called **integrable**, with integral \( I \), if for every \( \epsilon > 0 \), we may find a \( \delta > 0 \) such that for every mesh \( \mathcal{P} \) whose mesh size is less than \( \delta \), we have

\[
|I - S| < \epsilon,
\]

where \( S \) is any Riemann sum associated to \( \mathcal{P} \).

If \( W \subset \mathbb{R}^3 \) is a bounded subset and \( f : W \rightarrow \mathbb{R} \) is a bounded function, then pick a box \( B \) containing \( W \) and extend \( f \) by zero to a function \( \tilde{f} : B \rightarrow \mathbb{R} \),

\[
\tilde{f}(x) = \begin{cases} 
  x & \text{if } x \in W \\
  0 & \text{otherwise}.
\end{cases}
\]

If \( \tilde{f} \) is integrable, then we write

\[
\iiint_{W} f(x, y, z) \, dx \, dy \, dz = \iiint_{B} \tilde{f}(x, y, z) \, dx \, dy \, dz.
\]

In particular

\[
\text{vol}(W) = \iiint_{W} dx \, dy \, dz.
\]

There are two pairs of results, which are much the same as the results for double integrals:
Proposition 24.4. Let \( W \subset \mathbb{R}^2 \) be a bounded subset and let \( f: W \rightarrow \mathbb{R} \) and \( g: W \rightarrow \mathbb{R} \) be two integrable functions. Let \( \lambda \) be a scalar.

Then

(1) \( f + g \) is integrable over \( W \) and
\[
\iiint_{W} f(x, y, z) + g(x, y, z) \, dx \, dy \, dz = \iiint_{W} f(x, y, z) \, dx \, dy \, dz + \iiint_{W} g(x, y, z) \, dx \, dy \, dz.
\]

(2) \( \lambda f \) is integrable over \( W \) and
\[
\iiint_{W} \lambda f(x, y, z) \, dx \, dy \, dz = \lambda \iiint_{W} f(x, y, z) \, dx \, dy \, dz.
\]

(3) If \( f(x, y, z) \leq g(x, y, z) \) for any \( (x, y, z) \in W \), then
\[
\iiint_{W} f(x, y, z) \, dx \, dy \, dz \leq \iiint_{W} g(x, y, z) \, dx \, dy \, dz.
\]

(4) \( |f| \) is integrable over \( W \) and
\[
|\iiint_{W} f(x, y, z) \, dx \, dy \, dz| \leq \iiint_{W} |f(x, y, z)| \, dx \, dy \, dz.
\]

Proposition 24.5. Let \( W = W_1 \cup W_2 \subset \mathbb{R}^3 \) be a bounded subset and let \( f: W \rightarrow \mathbb{R} \) be a bounded function.

If \( f \) is integrable over \( W_1 \) and over \( W_2 \), then \( f \) is integrable over \( W \) and \( W_1 \cap W_2 \), and we have
\[
\iiint_{W} f(x, y, z) \, dx \, dy \, dz = \iiint_{W_1} f(x, y, z) \, dx \, dy \, dz + \iiint_{W_2} f(x, y, z) \, dx \, dy \, dz
\]
\[
- \iiint_{W_1 \cap W_2} f(x, y, z) \, dx \, dy \, dz.
\]

Definition 24.6. Define three maps
\[
\pi_{ij}: \mathbb{R}^3 \rightarrow \mathbb{R}^2,
\]
by projection onto the \( i \)th and \( j \)th coordinate.

In coordinates, we have
\[
\pi_{12}(x, y, z) = (x, y), \quad \pi_{23}(x, y, z) = (y, z), \quad \text{and} \quad \pi_{13}(x, y, z) = (x, z).
\]

For example, if we start with a solid pyramid and project onto the \( xy \)-plane, the image is a square, but it project onto the \( xz \)-plane, the image is a triangle. Similarly onto the \( yz \)-plane.

Definition 24.7. A bounded subset \( W \subset \mathbb{R}^3 \) is an \textbf{elementary subset} if it is one of four types:

\textbf{Type 1}: \( D = \pi_{12}(W) \) is an elementary region and
\[
W = \{ (x, y, z) \in \mathbb{R}^2 \mid (x, y) \in D, \epsilon(x, y) \leq z \leq \phi(x, y) \}.
\]
where $\epsilon: D \to \mathbb{R}$ and $\phi: D \to \mathbb{R}$ are continuous functions.

**Type 2:** $D = \pi_{23}(W)$ is an elementary region and

$$W = \{(x,y,z) \in \mathbb{R}^2 \mid (y,z) \in D, \alpha(y,z) \leq x \leq \beta(y,z)\},$$

where $\alpha: D \to \mathbb{R}$ and $\beta: D \to \mathbb{R}$ are continuous functions.

**Type 3:** $D = \pi_{13}(W)$ is an elementary region and

$$W = \{(x,y,z) \in \mathbb{R}^2 \mid (x,z) \in D, \gamma(x,z) \leq y \leq \delta(x,z)\},$$

where $\gamma: D \to \mathbb{R}$ and $\delta: D \to \mathbb{R}$ are continuous functions.

**Type 4:** $W$ is of type 1, 2 and 3.

The solid pyramid is of type 4.

**Theorem 24.8.** Let $W \subset \mathbb{R}^3$ be an elementary region and let $f: W \to \mathbb{R}$ be a continuous function.

Then

(1) If $W$ is of type 1, then

$$\iiint_W f(x,y,z) \, dx \, dy \, dz = \iint_{\pi_{12}(W)} \left( \int_{\epsilon(x,y)}^{\phi(x,y)} f(x,y,z) \, dz \right) \, dx \, dy.$$

(2) If $W$ is of type 2, then

$$\iiint_W f(x,y,z) \, dx \, dy \, dz = \iint_{\pi_{23}(W)} \left( \int_{\alpha(y,z)}^{\beta(y,z)} f(x,y,z) \, dx \right) \, dy \, dz.$$

(3) If $W$ is of type 3, then

$$\iiint_W f(x,y,z) \, dx \, dy \, dz = \iint_{\pi_{13}(W)} \left( \int_{\gamma(x,z)}^{\delta(x,z)} f(x,y,z) \, dy \right) \, dx \, dz.$$

Let’s figure out the volume of the solid ellipsoid:

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1\}.$$
This is an elementary region of type 4.

\[
\text{vol}(W) = \iiint_W \, dx \, dy \, dz \\
= \int_{-a}^{a} \left( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} \left( \int_{-c\sqrt{1-(\frac{y}{b})^2}}^{c\sqrt{1-(\frac{y}{b})^2}} dz \right) dy \right) \, dx \\
= \int_{-a}^{a} \left( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} 2c \sqrt{1 - \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2} dy \right) \, dx \\
= 2c \int_{-a}^{a} \left( \int_{-b\sqrt{1-(\frac{x}{a})^2}}^{b\sqrt{1-(\frac{x}{a})^2}} \frac{b^2}{1 - \left( \frac{x}{a} \right)^2 - y^2} dy \right) \, dx \\
= \frac{\pi c}{b} \int_{-a}^{a} \frac{b^2}{1 - \left( \frac{x}{a} \right)^2} \, dx \\
= \pi bc \int_{-a}^{a} 1 - \left( \frac{x}{a} \right)^2 \, dx \\
= \pi bc \left[ x - \frac{x^3}{3a^2} \right]_{-a}^{a} \\
= \pi bc \left( 2a - 2\frac{a^3}{3a^2} \right) \\
= \frac{4\pi}{3} abc.
\]