23. Inclusion-Exclusion

**Proposition 23.1.** Let \( D = D_1 \cup D_2 \) be a bounded region and let \( f: D \rightarrow \mathbb{R} \) be a function.

If \( f \) is integrable over \( D_1 \) and over \( D_2 \), then \( f \) is integrable over \( D \) and \( D_1 \cap D_2 \), and we have

\[
\int\int_D f(x, y) \, dx \, dy = \int\int_{D_1} f(x, y) \, dx \, dy + \int\int_{D_2} f(x, y) \, dx \, dy - \int\int_{D_1 \cap D_2} f(x, y) \, dx \, dy.
\]

**Example 23.2.** Let

\[
D = \{ (x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 9 \}.
\]

Then \( D \) is not an elementary region. Let

\[
D_1 = \{ (x, y) \in D \mid y \geq 0 \} \quad \text{and} \quad D_2 = \{ (x, y) \in D \mid y \leq 0 \}.
\]

Then \( D_1 \) and \( D_2 \) are both of type 1.

If \( f \) is continuous, then \( f \) is integrable over \( D \) and \( D_1 \cap D_2 \). In fact

\[
D_1 \cap D_2 = L \cup R = \{ (x, y) \in \mathbb{R}^2 \mid -3 \leq x \leq -1, 0 \leq y \leq 0 \}
\]
\[
\quad \cup \{ (x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 3, 0 \leq y \leq 0 \}.
\]

Now \( L \) and \( R \) are elementary regions. We have

\[
\int\int_R f(x, y) \, dx \, dy = \int_{-1}^{0} \left( \int_{0}^{0} f(x, y) \, dy \right) \, dx = 0.
\]

Therefore, by symmetry,

\[
\int\int_L f(x, y) \, dx \, dy = \int\int_R f(x, y) \, dx \, dy = 0
\]

and so

\[
\int\int_D f(x, y) \, dx \, dy = \int\int_{D_1} f(x, y) \, dx \, dy + \int\int_{D_2} f(x, y) \, dx \, dy.
\]
To integrate $f$ over $D_1$, break $D_1$ into three parts.

\[
\iint_{D_1} f(x,y) \, dx \, dy = \int_{-3}^{3} \left( \int_{\gamma(x)}^{\delta(x)} f(x,y) \, dy \right) \, dx
\]
\[
= \int_{-3}^{-1} \left( \int_{0}^{\sqrt{9-x^2}} f(x,y) \, dy \right) \, dx
\]
\[
+ \int_{-1}^{1} \left( \int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} f(x,y) \, dy \right) \, dx
\]
\[
+ \int_{1}^{3} \left( \int_{0}^{\sqrt{9-x^2}} f(x,y) \, dy \right) \, dx.
\]

One can do something similar for $D_2$.

**Example 23.3.** Suppose we are given that

\[
\iint_{D} f(x,y) \, dx \, dy = \int_{0}^{1} \left( \int_{x/2}^{2y} f(x,y) \, dx \right) \, dy.
\]

What is the region $D$?

It is the region bounded by the two lines $y = x$ and $x = 2y$ and between the two lines $y = 0$ and $y = 1$.

Change order of integration:

\[
\iint_{D} f(x,y) \, dx \, dy = \int_{0}^{1} \left( \int_{x/2}^{2y} f(x,y) \, dx \right) \, dy + \int_{1}^{2} \left( \int_{x/2}^{1} f(x,y) \, dx \right) \, dy.
\]

**Example 23.4.** Calculate the volume of a solid ball of radius $a$. Let

\[
B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq a^2\}.
\]

We want the volume of $B$. Break into two pieces. Let

\[
B^+ = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq a^2, z \geq 0\}.
\]

Let

\[
D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}.
\]

Then $B^+$ is bounded by the $xy$-plane and the graph of the function

\[f: D \longrightarrow \mathbb{R},\]

given by

\[f(x,y) = \sqrt{a^2 - x^2 - y^2}.\]
It follows that
\[
\text{vol}(B^+) = \int\int_D \sqrt{a^2 - x^2 - y^2} \, dy \, dx
\]
\[
= \int_{-a}^a \left( \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \right) \, dx
\]
\[
= \int_{-a}^a \left( \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \sqrt{1 - \frac{y^2}{a^2 - x^2}} \sqrt{a^2 - x^2} \, dy \right) \, dx.
\]

Now let’s make the substitution
\[
t = \frac{y}{\sqrt{a^2 - x^2}} \quad \text{so that} \quad dt = \frac{dy}{\sqrt{a^2 - x^2}}.
\]
\[
\text{vol}(B^+) = \int_{-a}^a \left( \int_{-1}^1 \sqrt{1 - t^2} (a^2 - x^2) \, dt \right) \, dx
\]
\[
= \int_{-a}^a (a^2 - x^2) \left( \int_{-1}^1 \sqrt{1 - t^2} \, dt \right) \, dx
\]
\[
= \int_{-a}^a (a^2 - x^2) \pi \frac{1}{2} \, dx
\]
\[
= \pi \left[ a^2 x - \frac{x^3}{3} \right]_{-a}^a
\]
\[
= \pi (a^3 - \frac{a^3}{3})
\]
\[
= \frac{2\pi a^3}{3}.
\]

Therefore, we get the expected answer
\[
\text{vol}(B) = 2 \text{vol}(B^+) = \frac{4\pi a^3}{3}.
\]

Example 23.5. Now consider the example of a cone whose base radius is \(a\) and whose height is \(b\). Put the central axis along the \(x\)-axis and
the base in the $yz$-plane. In the $xy$-plane we get an equilateral triangle of height $b$ and base $2a$. If we view this as a region of type 1, we have
\[ \gamma(x) = -a \left(1 - \frac{x}{b}\right) \quad \text{and} \quad \delta(x) = a \left(1 - \frac{x}{b}\right). \]
We want to integrate the function
\[ f: D \longrightarrow \mathbb{R}, \]
given by
\[ f(x, y) = \sqrt{a^2 \left(1 - \frac{x}{b}\right)^2 - y^2}. \]
So half of the volume of the cone is
\[ \int_0^b \left( \int_{-a(1-\frac{x}{b})}^{a(1-\frac{x}{b})} \sqrt{a^2 \left(1 - \frac{x}{b}\right)^2 - y^2} \, dy \right) \, dx = \frac{\pi}{2} \int_0^b a^2 \left(1 - \frac{x}{b}\right)^2 \, dx \]
\[ = \frac{\pi a^2}{2} \int_0^b 1 - \frac{2x}{b} + \frac{x^2}{b^2} \, dx \]
\[ = \frac{\pi a^2}{2} \left[ x - \frac{2x^2}{b} + \frac{x^3}{3b^2} \right]_0^b \]
\[ = \frac{1}{6} (\pi a^2 b). \]
Therefore the volume is
\[ \frac{1}{3} (\pi a^2 b). \]