23. Inclusion-Exclusion

Proposition 23.1. *Let $D = D_1 \cup D_2$ be a bounded region and let $f \colon D \longrightarrow \mathbb{R}$ be a function.*

If f is integrable over D_1 and over D_2, then f is integrable over D and and $D_1 \cap D_2$, and we have

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_1\cap D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Example 23.2. *Let*

$$D = \{\,(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 9\,\}.$$

Then D is not an elementary region. Let

$$D_1 = \{\,(x,y) \in D \mid y \geq 0\,\} \qquad \text{and} \qquad D_2 = \{\,(x,y) \in D \mid y \leq 0\,\}.$$

Then D_1 and D_2 are both of type 1.

If f is continuous, then f is integrable over D and $D_1 \cap D_2$. In fact

$$\begin{aligned} D_1 \cap D_2 = L \cup R = \{\,(x,y) \in \mathbb{R}^2 \mid -3 \leq x \leq -1, 0 \leq y \leq 0\,\} \\ \cup \{\,(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 3, 0 \leq y \leq 0\,\}. \end{aligned}$$

Now L and R are elementary regions. We have

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^3 \left(\int_0^0 f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = 0.$$

Therefore, by symmetry,

$$\iint_L f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

and so

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

To integrate f over D_1, break D_1 into three parts.

$$\begin{aligned}\iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-3}^{3}\left(\int_{\gamma(x)}^{\delta(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-3}^{-1}\left(\int_{0}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\ &+ \int_{-1}^{1}\left(\int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\ &+ \int_{1}^{3}\left(\int_{0}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.\end{aligned}$$

One can do something similar for D_2.

Example 23.3. *Suppose we are given that*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left(\int_y^{2y} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

What is the region D?

It is the region bounded by the two lines $y = x$ and $x = 2y$ and between the two lines $y = 0$ and $y = 1$.

Change order of integration:

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left(\int_{x/2}^{x} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y + \int_1^2\left(\int_{x/2}^{1} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

Example 23.4. *Calculate the volume of a solid ball of radius a. Let*

$$B = \{\,(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq a^2\,\}.$$

We want the volume of B. Break into two pieces. Let

$$B^+ = \{\,(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \leq a^2, z \geq 0\,\}.$$

Let

$$D = \{\,(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq a^2\,\}.$$

Then B^+ is bounded by the xy-plane and the graph of the function

$$f\colon D \longrightarrow \mathbb{R},$$

given by

$$f(x,y) = \sqrt{a^2 - x^2 - y^2}.$$

It follows that

$$
\begin{aligned}
\operatorname{vol}(B^+) &= \iint_D \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{1 - \frac{y^2}{a^2 - x^2}} \sqrt{a^2 - x^2}\,\mathrm{d}y \right) \mathrm{d}x.
\end{aligned}
$$

Now let's make the substitution

$$
t = \frac{y}{\sqrt{a^2 - x^2}} \qquad \textit{so that} \qquad dt = \frac{dy}{\sqrt{a^2 - x^2}}.
$$

$$
\begin{aligned}
\operatorname{vol}(B^+) &= \int_{-a}^{a} \left(\int_{-1}^{1} \sqrt{1 - t^2}(a^2 - x^2)\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_{-a}^{a} (a^2 - x^2) \left(\int_{-1}^{1} \sqrt{1 - t^2}\,\mathrm{d}t \right) \mathrm{d}x
\end{aligned}
$$

Now let's make the substitution

$$
t = \sin u \qquad \textit{so that} \qquad dt = \cos u\,du.
$$

$$
\begin{aligned}
\operatorname{vol}(B^+) &= \int_{-a}^{a} (a^2 - x^2) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u \right) \mathrm{d}x \\
&= \int_{-a}^{a} (a^2 - x^2) \frac{\pi}{2}\,\mathrm{d}x \\
&= \frac{\pi}{2} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a} \\
&= \pi(a^3 - \frac{a^3}{3}) \\
&= \frac{2\pi a^3}{3}.
\end{aligned}
$$

Therefore, we get the expected answer

$$
\operatorname{vol}(B) = 2\operatorname{vol}(B^+) = \frac{4\pi a^3}{3}.
$$

Example 23.5. *Now consider the example of a cone whose base radius is a and whose height is b. Put the central axis along the x-axis and* *the base in the yz-plane. In the xy-plane we get an equilateral triangle of height b and base $2a$. If we view this as a region of type 1, we have*

$$\gamma(x) = -a\left(1 - \frac{x}{b}\right) \qquad \textit{and} \qquad \delta(x) = a\left(1 - \frac{x}{b}\right).$$

We want to integrate the function

$$f\colon D \longrightarrow \mathbb{R},$$

given by

$$f(x, y) = \sqrt{a^2\left(1 - \frac{x}{b}\right)^2 - y^2}.$$

So half of the volume of the cone is

$$\begin{aligned}
\int_0^b \left(\int_{-a(1-\frac{x}{b})}^{a(1-\frac{x}{b})} \sqrt{a^2\left(1 - \frac{x}{b}\right)^2 - y^2}\,\mathrm{d}y\right)\mathrm{d}x &= \frac{\pi}{2}\int_0^b a^2\left(1 - \frac{x}{b}\right)^2\,\mathrm{d}x \\
&= \frac{\pi a^2}{2}\int_0^b 1 - \frac{2x}{b} + \frac{x^2}{b^2}\,\mathrm{d}x \\
&= \frac{\pi a^2}{2}\left[x - \frac{x^2}{b} + \frac{x^3}{3b^2}\right]_0^b \\
&= \frac{1}{6}(\pi a^2 b).
\end{aligned}$$

Therefore the volume is

$$\frac{1}{3}(\pi a^2 b).$$