22. Double integrals

Definition 22.1. Let \(R = [a, b] \times [c, d] \subset \mathbb{R}^2\) be a rectangle in the plane. A partition \(\mathcal{P}\) of \(R\) is a pair of sequences:

\[
a = x_0 < x_1 < \cdots < x_n = b \\
c = y_0 < y_1 < \cdots < y_n = d.
\]

The mesh of \(\mathcal{P}\) is

\[
m(\mathcal{P}) = \max\{x_i - x_{i-1}, y_i - y_{i-1} \mid 1 \leq i \leq k\}.
\]

Now suppose we are given a function

\[f : R \rightarrow \mathbb{R}\]

Pick

\[
\bar{c}_{ij} \in R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].
\]

Definition 22.2. The sum

\[
S = \sum_{i=1}^{n} \sum_{j=1}^{n} f(\bar{c}_{ij})(x_i - x_{i-1})(y_j - y_{j-1}),
\]

is called a Riemann sum.

We will use the short hand notation

\[
\Delta x_i = x_i - x_{i-1} \quad \text{and} \quad \Delta y_j = y_j - y_{j-1}.
\]

Definition 22.3. The function \(f : R \rightarrow \mathbb{R}\) is called integrable, with integral \(I\), if for every \(\epsilon > 0\), we may find a \(\delta > 0\) such that for every mesh \(\mathcal{P}\) whose mesh size is less than \(\delta\), we have

\[
|I - S| < \epsilon,
\]

where \(S\) is any Riemann sum associated to \(\mathcal{P}\).

We write

\[
\iint_{R} f(x, y) \, dx \, dy = I,
\]

to mean that \(f\) is integrable with integral \(I\).

We use a sneaky trick to integrate over regions other than rectangles. Suppose that \(D\) is a bounded subset of the plane. Then we can find a rectangle \(R\) which completely contains \(D\).

Definition 22.4. The indicator function of \(D \subset R\) is the function

\[i_D : R \rightarrow \mathbb{R},\]
given by
\[ i_D(x) = \begin{cases} 
1 & \text{if } x \in D \\
0 & \text{if } x \notin D.
\end{cases} \]

If \( i_D \) is integrable, then we say that the \textbf{area of} \( D \) is the integral
\[ \iint_R i_D \, dx \, dy. \]

If \( i_D \) is not integrable, then \( D \) does not have an area.

**Example 22.5.** Let
\[ D = \{ (x, y) \in [0, 1] \times [0, 1] \mid x, y \in \mathbb{Q} \}. \]
Then \( D \) does not have an area.

**Definition 22.6.** If \( f: D \to \mathbb{R} \) is a function and \( D \) is bounded, then pick \( D \subset R \subset \mathbb{R}^2 \) a rectangle. Define
\[ \tilde{f}: R \to \mathbb{R}, \]
by the rule
\[ \tilde{f}(x) = \begin{cases} 
f(x) & \text{if } x \in D \\
0 & \text{otherwise.}
\end{cases} \]

We say that \( f \) is \textbf{integrable} over \( D \) if \( \tilde{f} \) is integrable over \( R \). In this case
\[ \iint_D f(x, y) \, dx \, dy = \iint_R \tilde{f}(x, y) \, dx \, dy. \]

**Proposition 22.7.** Let \( D \subset \mathbb{R}^2 \) be a bounded subset and let \( f: D \to \mathbb{R} \) and \( g: D \to \mathbb{R} \) be two integrable functions. Let \( \lambda \) be a scalar.
Then
\begin{enumerate}
\item \( f + g \) is integrable over \( D \) and
\[ \iint_D f(x, y) + g(x, y) \, dx \, dy = \iint_D f(x, y) \, dx \, dy + \iint_D g(x, y) \, dx \, dy. \]
\item \( \lambda f \) is integrable over \( D \) and
\[ \iint_D \lambda f(x, y) \, dx \, dy = \lambda \iint_D f(x, y) \, dx \, dy. \]
\item If \( f(x, y) \leq g(x, y) \) for any \( (x, y) \in D \), then
\[ \iint_D f(x, y) \, dx \, dy \leq \iint_D g(x, y) \, dx \, dy. \]
\end{enumerate}
(4) $|f|$ is integrable over $D$ and

$$|\iint_D f(x, y) \, dx \, dy| \leq \iint_D |f(x, y)| \, dx \, dy.$$ 

It is straightforward to integrate continuous functions over regions of three special types:

**Definition 22.8.** A bounded subset $D \subset \mathbb{R}^2$ is an *elementary region* if it is one of three types:

**Type 1:**

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \gamma(x) \leq y \leq \delta(x)\},$$

where $\gamma: [a, b] \rightarrow \mathbb{R}$ and $\delta: [a, b] \rightarrow \mathbb{R}$ are continuous functions.

**Type 2:**

$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \alpha(y) \leq x \leq \beta(y)\},$$

where $\alpha: [c, d] \rightarrow \mathbb{R}$ and $\beta: [c, d] \rightarrow \mathbb{R}$ are continuous functions.

**Type 3:** $D$ is both type 1 and 2.

**Theorem 22.9.** Let $D \subset \mathbb{R}^2$ be an elementary region and let $f: D \rightarrow \mathbb{R}$ be a continuous function.

Then

(1) If $D$ is of type 1, then

$$\iint_D f(x, y) \, dx \, dy = \int_a^b \left( \int_{\gamma(x)}^{\delta(x)} f(x, y) \, dy \right) \, dx.$$ 

(2) If $D$ if of type 2, then

$$\iint_D f(x, y) \, dx \, dy = \int_c^d \left( \int_{\alpha(y)}^{\beta(y)} f(x, y) \, dx \right) \, dy.$$ 

**Example 22.10.** Let $D$ be the region bounded by the lines $x = 0$, $y = 4$ and the parabola $y = x^2$. Let $f: D \rightarrow \mathbb{R}$ be the function given by $f(x, y) = x^2 + y^2$.  

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If we view \( D \) as a region of type 1, then we get
\[
\iint_D f(x, y) \, dx \, dy = \int_0^2 \left( \int_{x^2}^4 x^2 + y^2 \, dy \right) \, dx
\]
\[
= \int_0^2 \left[ x^2 y + \frac{y^3}{3} \right]_{x^2}^4 \, dx
\]
\[
= \int_0^2 4x^2 + \frac{2^6}{3} - x^4 - \frac{x^6}{3} \, dx
\]
\[
= \left[ \frac{4x^3}{3} + \frac{2^6x}{3} - \frac{x^5}{5} - \frac{x^7}{3 \cdot 7} \right]_0^2
\]
\[
= \frac{2^5}{3} + \frac{2^7}{3} - \frac{2^5}{5} - \frac{2^7}{3 \cdot 7}
\]
\[
= \frac{2^6}{5} + \frac{2^8}{7}
\]
\[
= 2^6 \left( \frac{1}{3 \cdot 5} + \frac{2^2}{7} \right).
\]
On the other hand, if we view \( D \) as a region of type 2, then we get
\[
\iint_D f(x, y) \, dx \, dy = \int_0^4 \left( \int_0^{\sqrt{y}} x^2 + y^2 \, dx \right) \, dy
\]
\[
= \int_0^4 \left[ \frac{x^3}{3} + xy^2 \right]^{\sqrt{y}}_0 \, dy
\]
\[
= \int_0^4 \frac{y^{3/2}}{3} + y^{5/2} \, dy
\]
\[
= \left[ \frac{2y^{5/2}}{3 \cdot 5} + \frac{2y^{7/2}}{7} \right]_0^4
\]
\[
= \frac{2^6}{3 \cdot 5} + \frac{2^8}{7}
\]
\[
= 2^6 \left( \frac{1}{3 \cdot 5} + \frac{2^2}{7} \right).
\]