21. Maxima and minima: II

To see how to maximise and minimise a function on the boundary, let’s consider a concrete example.

Let
\[ K = \{ (x, y) \mid x^2 + y^2 \leq 2 \} \]

Then \( K \) is compact. Let
\[ f: K \rightarrow \mathbb{R}, \]
be the function \( f(x, y) = xy \). Then \( f \) is continuous and so \( f \) achieves its maximum and minimum.

I. Let’s first consider the interior points. Then
\[ \nabla f(x, y) = (y, x), \]
so that \((0, 0)\) is the only critical point. The Hessian of \( f \) is
\[ Hf(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

\( d_1 = 0 \) and \( d_2 = -1 \neq 0 \) so that \((0, 0)\) is a saddle point.

It follows that the maxima and minima of \( f \) are on the boundary, that is, the set of points
\[ C = \{ (x, y) \mid x^2 + y^2 = 2 \}. \]

II. Let \( g: \mathbb{R}^2 \rightarrow \mathbb{R} \) be the function \( g(x, y) = x^2 + y^2 \). Then the circle \( C \) is a level curve of \( g \). The original problem asks to maximise and minimise
\[ f(x, y) = xy \quad \text{subject to} \quad g(x, y) = x^2 + y^2 = 2. \]

One way to proceed is to use the second equation to eliminate a variable. The method of Lagrange multipliers does exactly the opposite. Instead of eliminating a variable we add one more variable, traditionally called \( \lambda \). So now let’s maximise and minimise
\[ h(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 2) = xy - \lambda(x^2 + y^2 - 2). \]

We find the critical points of \( h(x, y, \lambda) \):
\[ \begin{align*}
  y &= 2\lambda x \\
  x &= 2\lambda y \\
  2 &= x^2 + y^2.
\end{align*} \]

First note that if \( x = 0 \) then \( y = 0 \) and \( x^2 + y^2 = 0 \neq 2 \), impossible. So \( x \neq 0 \). Similarly one can check that \( y \neq 0 \) and \( \lambda \neq 0 \). Divide the
first equation by the second:

\[ \frac{y}{x} = \frac{x}{y}, \]

so that \( y^2 = x^2 \). As \( x^2 + y^2 = 2 \) it follows that \( x^2 = y^2 = 1 \). So \( x = \pm 1 \) and \( y = \pm 1 \). This gives four potential points \((1, 1), (-1, 1), (1, -1), (-1, -1)\). Then the maximum value of \( f \) is 1, and this occurs at the first and the last point. The minimum value of \( f \) is \(-1\), and this occurs at the second and the third point.

One can also try to parametrise the boundary:

\[ \vec{r}(t) = \sqrt{2}(\cos t, \sin t). \]

So we maximise the composition

\[ h: [0, 2\pi] \to \mathbb{R}, \]

where \( h(t) = 2 \cos t \sin t \). As \( I = [0, 2\pi] \) is compact, \( h \) has a maximum and minimum on \( I \). When \( h'(t) = 0 \), we get

\[ \cos^2 t - \sin^2 t = 0. \]

Note that the LHS is \( \cos 2t \), so we want

\[ \cos 2t = 0. \]

It follows that \( 2t = \pi/2 + 2m\pi \), so that

\[ t = \pi/4, \quad 3\pi/4, \quad 5\pi/4, \quad \text{and} \quad 7\pi/4. \]

These give the four points we had before.

What is the closest point to the origin on the surface

\[ F = \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, xyz = p \}? \]

So we want to minimise the distance to the origin on \( F \). The first trick is to minimise the square of the distance. In other words, we are trying to minimise \( f(x, y, z) = x^2 + y^2 + z^2 \) on the surface

\[ F = \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, z \geq 0, xyz = p \}. \]

In words, given three numbers \( x \geq 0, y \geq 0 \) and \( z \geq 0 \) whose product is \( p > 0 \), what is the minimum value of \( x^2 + y^2 + z^2 \)?

Now \( F \) is closed but it is not bounded, so it is not even clear that the minimum exists.

Let’s use the method of Lagrange multipliers. Let

\[ h: \mathbb{R}^4 \to \mathbb{R}, \]

be the function

\[ h(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xyz - p). \]
We look for the critical points of $h$:

\[
\begin{align*}
2x &= \lambda yz \\
2y &= \lambda xz \\
2z &= \lambda xy \\
p &= xyz.
\end{align*}
\]

Once again, it is not possible for any of the variables to be zero. Taking the product of the first three equations, we get

\[8xyz = \lambda^3 (x^2 y^2 z^2).\]

So, dividing by $xyz$ and using the last equation, we get

\[8 = \lambda^3 p,\]

that is

\[\lambda = \frac{2}{p^{1/3}}.\]

Taking the product of the first two equations, and dividing by $xy$, we get

\[4 = \lambda^2 z^2,\]

so that

\[z = p^{1/3}.\]

So $h(x, y, z, \lambda)$ has a critical point at

\[(x, y, z, \lambda) = (p^{1/3}, p^{1/3}, p^{1/3}, \frac{2}{p^{1/3}}).\]

We check that the point

\[(x, y, z) = (p^{1/3}, p^{1/3}, p^{1/3}),\]

is a minimum of $x^2 + y^2 + z^2$ subject to the constraint $xyz = p$. At this point the sum of the squares is

\[3p^{2/3}.\]

Suppose that $x \geq \sqrt[3]{3}p^{1/3}$. Then the sum of the squares is at least $3p^{2/3}$. Similarly if $y \geq \sqrt[3]{3}p^{1/3}$ or $z \geq \sqrt[3]{3}p^{1/3}$. On the other hand, the set

\[K = \{(x, y, z) \in \mathbb{R}^3 | x \in [0, \sqrt[3]{3}p^{1/3}], y \in [0, \sqrt[3]{3}p^{1/3}], z \in [0, \sqrt[3]{3}p^{1/3}], xyz = p\},\]

is closed and bounded, so that $f$ achieves it minimum on this set, which we have already decided is at

\[(x, y, z) = (p^{1/3}, p^{1/3}, p^{1/3}),\]

since $f$ is larger on the boundary. Putting all of this together, the point

\[(x, y, z) = (p^{1/3}, p^{1/3}, p^{1/3}),\]
is a point where the sum of the squares is a minimum.

Here is another such problem. Find the closest point to the origin which also belongs to the cone

\[ x^2 + y^2 = z^2, \]

and to the plane

\[ x + y + z = 3. \]

As before, we minimise \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to \( g_1(x, y, z) = x^2 + y^2 - z^2 = 0 \) and \( g_2(x, y, z) = x + y + z = 3 \). Introduce a new function, with two new variables \( \lambda_1 \) and \( \lambda_2 \),

\[ h : \mathbb{R}^5 \rightarrow \mathbb{R}, \]

given by

\[ h(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z) \]
\[ = x^2 + y^2 + z^2 - \lambda_1(x^2 + y^2 - z^2) - \lambda_2(x + y + z - 3). \]

We find the critical points of \( h \):

\[
\begin{align*}
2x &= 2\lambda_1 x + \lambda_2 \\
2y &= 2\lambda_1 y + \lambda_2 \\
2z &= -2\lambda_1 z + \lambda_2 \\
z^2 &= x^2 + y^2 \\
3 &= x + y + z.
\end{align*}
\]

Suppose we substract the first equation from the second:

\[ y - x = \lambda_1(y - x). \]

So either \( x = y \) or \( \lambda_1 = 1 \). Suppose \( x \neq y \). Then \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). In this case \( z = -z \), so that \( z = 0 \). But then \( x^2 + y^2 = 0 \) and so \( x = y = 0 \), which is not possible.

It follows that \( x = y \), in which case \( z = \pm \sqrt{2}x \) and

\[ (2 \pm \sqrt{2})x = 3. \]

So

\[ x = \frac{3}{2 \pm \sqrt{2}} = \frac{3(2 \mp \sqrt{2})}{2}. \]

This gives us two critical points:

\[ P_1 = \left( \frac{3(2 - \sqrt{2})}{2}, \frac{3(2 - \sqrt{2})}{2}, \frac{3\sqrt{2}(2 - \sqrt{2})}{2} \right) \]
\[ P_2 = \left( \frac{3(2 + \sqrt{2})}{2}, \frac{3(2 + \sqrt{2})}{2}, \frac{-3\sqrt{2}(2 - \sqrt{2})}{2} \right). \]

Of the two, clearly the first is closest to the origin.
To finish, we had better show that this point is the closest to the origin on the whole locus

\[ F = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, x + y + z = 3 \}. \]

Let

\[ K = \{ (x, y, z) \in F \mid x^2 + y^2 + z^2 \leq 25 \}. \]

Then \( K \) is closed and bounded, whence compact. So \( f \) achieves its minimum somewhere on \( K \), and so it must achieve its minimum at \( P = P_1 \). Clearly outside \( f \) is at least 25 on \( F \setminus K \), and so \( f \) is a minimum at \( P_1 \) on the whole of \( F \).