2. Dot product

Definition 2.1. Let \( \vec{v} = (v_1, v_2, v_3) \) and \( \vec{w} = (w_1, w_2, w_3) \) be two vectors in \( \mathbb{R}^3 \). The dot product of \( \vec{v} \) and \( \vec{w} \), denoted \( \vec{v} \cdot \vec{w} \), is the scalar \( v_1w_1 + v_2w_2 + v_3w_3 \).

Example 2.2. The dot product of \( \vec{v} = (1, -2, -1) \) and \( \vec{w} = (2, 1, -3) \) is
\[
1 \cdot 2 + (-2) \cdot 1 + (-1) \cdot (-3) = 2 - 2 + 3 = 3.
\]

Lemma 2.3. Let \( \vec{u}, \vec{v}, \) and \( \vec{w} \) be three vectors in \( \mathbb{R}^3 \) and let \( \lambda \) be a scalar.

(1) \( (\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \).

(2) \( \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \).

(3) \( (\lambda \vec{v}) \cdot \vec{w} = \lambda (\vec{v} \cdot \vec{w}) \).

(4) \( \vec{v} \cdot \vec{v} = 0 \) if and only if \( \vec{v} = \vec{0} \).

Proof. (1–3) are straightforward.

To see (4), first note that one direction is clear. If \( \vec{v} = \vec{0} \), then \( \vec{v} \cdot \vec{v} = 0 \). For the other direction, suppose that \( \vec{v} \cdot \vec{v} = 0 \). Then \( v_1^2 + v_2^2 + v_3^2 = 0 \). Now the square of a real number is non-negative and if a sum of non-negative numbers is zero, then each term must be zero. It follows that \( v_1 = v_2 = v_3 = 0 \) and so \( \vec{v} = \vec{0} \). \( \square \)

Definition 2.4. If \( \vec{v} \in \mathbb{R}^3 \), then the norm or length of \( \vec{v} = (v_1, v_2, v_3) \) is the scalar
\[
\|v\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}.
\]

It is interesting to note that if you know the norm, then you can calculate the dot product:
\[
(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}
\]
\[
(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}.
\]

Subtracting and dividing by 4 we get
\[
\vec{v} \cdot \vec{w} = \frac{1}{4} (((\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) - (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}))
\]
\[
= \frac{1}{4} (\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2).
\]

Given two non-zero vectors \( \vec{v} \) and \( \vec{w} \) in space, note that we can define the angle \( \theta \) between \( \vec{v} \) and \( \vec{w} \). \( \vec{v} \) and \( \vec{w} \) lie in at least one plane (which is in fact unique, unless \( \vec{v} \) and \( \vec{w} \) are parallel). Now just measure the angle \( \theta \) between the \( \vec{v} \) and \( \vec{w} \) in this plane. By convention we always take \( 0 \leq \theta \leq \pi \).
**Theorem 2.5.** If $\vec{v}$ and $\vec{w}$ are any two vectors in $\mathbb{R}^3$, then
\[ \vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta. \]

**Proof.** If $\vec{v}$ is the zero vector, then both sides are equal to zero, so that they are equal to each other and the formula holds (note though, that in this case the angle $\theta$ is not determined).

By symmetry, we may assume that $\vec{v}$ and $\vec{w}$ are both non-zero. Let $\vec{u} = \vec{w} - \vec{v}$ and apply the law of cosines to the triangle with sides parallel to $\vec{u}$, $\vec{v}$ and $\vec{w}$:
\[ \|\vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos \theta. \]

We have already seen that the LHS of this equation expands to
\[ \vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = \|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2. \]

Cancelling the common terms $\|\vec{v}\|^2$ and $\|\vec{w}\|^2$ from both sides, and dividing by 2, we get the desired formula. □

We can use (2.5) to find the angle between two vectors:

**Example 2.6.** Let $\vec{v} = -\hat{i} + \hat{k}$ and $\vec{w} = \hat{i} + \hat{j}$. Then
\[ -1 = \vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos \theta = 2 \cos \theta. \]

Therefore $\cos \theta = -1/2$ and so $\theta = 2\pi/3$.

**Definition 2.7.** We say that two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.

**Remark 2.8.** If neither $\vec{v}$ nor $\vec{w}$ are the zero vector, and $\vec{v} \cdot \vec{w} = 0$ then the angle between $\vec{v}$ and $\vec{w}$ is a right angle. Our convention is that the zero vector is orthogonal to every vector.

**Example 2.9.** $\hat{i}$, $\hat{j}$ and $\hat{k}$ are pairwise orthogonal.

Given two vectors $\vec{v}$ and $\vec{w}$, we can project $\vec{v}$ onto $\vec{w}$. The resulting vector is called the projection of $\vec{v}$ onto $\vec{w}$ and is denoted $\text{proj}_\vec{w} \vec{v}$. For example, if $\vec{F}$ is a force and $\vec{w}$ is a direction, then the projection of $\vec{F}$ onto $\vec{w}$ is the force in the direction of $\vec{w}$.

As $\text{proj}_\vec{w} \vec{v}$ is parallel to $\vec{w}$, we have
\[ \text{proj}_\vec{w} \vec{v} = \lambda \vec{w}, \]
for some scalar $\lambda$. Let’s determine $\lambda$. Let’s deal with the case that $\lambda \geq 0$ (so that the angle $\theta$ between $\vec{v}$ and $\vec{w}$ is between 0 and $\pi/2$). If we take the norm of both sides, we get
\[ \|\text{proj}_\vec{w} \vec{v}\| = \|\lambda \vec{w}\| = \lambda \|\vec{w}\|, \]
(note that $\lambda \geq 0$), so that

$$\lambda = \frac{||\text{proj}_{\vec{w}} \vec{v}||}{||\vec{w}||}.$$  

But

$$\cos \theta = \frac{||\text{proj}_{\vec{w}} \vec{v}||}{||\vec{v}||},$$  

so that

$$||\text{proj}_{\vec{w}} \vec{v}|| = ||\vec{v}|| \cos \theta.$$  

Putting all of this together we get

$$\lambda = \frac{||\vec{v}|| \cos \theta}{||\vec{w}||} = \frac{||\vec{v}|| ||\vec{w}|| \cos \theta}{||\vec{w}||^2} = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2}.$$  

There are a number of ways to deal with the case when $\lambda < 0$ (so that $\theta > \pi/2$). One can carry out a similar analysis to the one given above. Here is another way. Note that the angle $\phi$ between $\vec{w}$ and $\vec{u} = -\vec{v}$ is equal to $\pi - \theta < \pi/2$. By what we already proved

$$\text{proj}_{\vec{w}} \vec{u} = \frac{\vec{u} \cdot \vec{w}}{||\vec{w}||^2} \vec{w}.$$  

But $\text{proj}_{\vec{w}} \vec{u} = -\text{proj}_{\vec{w}} \vec{v}$ and $\vec{u} \cdot \vec{w} = -\vec{v} \cdot \vec{w}$, so we get the same formula in the end. To summarise:

**Theorem 2.10.** If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^3$, where $\vec{w}$ is not zero, then

$$\text{proj}_{\vec{w}} \vec{v} = \left( \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2} \right) \vec{w}.$$  

**Example 2.11.** Find the distance $d$ between the line $l$ containing the points $P_1 = (1, -1, 2)$ and $P_2 = (4, 1, 0)$ and the point $Q = (3, 2, 4)$.

Suppose that $R$ is the closest point on the line $l$ to the point $Q$. Note that $\overrightarrow{QR}$ is orthogonal to the direction $\overrightarrow{P_1P_2}$ of the line. So we want the length of the vector $\overrightarrow{P_1Q} - \text{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q}$, that is, we want

$$d = ||\overrightarrow{P_1Q} - \text{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q}||.$$  

Now

$$\overrightarrow{P_1Q} = (2, 3, 2) \quad \text{and} \quad \overrightarrow{P_1P_2} = (3, 2, -2).$$
We have
\[ \|P_1P_2\|^2 = 3^2 + 2^2 + 2^2 = 17 \quad \text{and} \quad \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q} = 6 + 6 - 4 = 8. \]

It follows that
\[ \text{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q} = \frac{8}{17} (3, 2, -2). \]

Subtracting, we get
\[ \overrightarrow{P_1Q} - \text{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q} = (2, 3, 2) - \frac{8}{17} (3, 2, -2) = \frac{1}{17} (10, 35, 50) = \frac{5}{17} (2, 7, 10). \]

Taking the length, we get
\[ \frac{5}{17} (2^2 + 7^2 + 10^2)^{1/2} \approx 3.64. \]

**Theorem 2.12.** The angle subtended on the circumference of a circle by a diameter of the circle is always a right angle.

**Proof.** Suppose that \( P \) and \( Q \) are the two endpoints of a diameter of the circle and that \( R \) is a point on the circumference. We want to show that the angle between \( \overrightarrow{PR} \) and \( \overrightarrow{QR} \) is a right angle.

Let \( O \) be the centre of the circle. Then
\[ \overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} \quad \text{and} \quad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}. \]

Note that \( \overrightarrow{QO} = -\overrightarrow{PO} \). Therefore
\[
\overrightarrow{PR} \cdot \overrightarrow{QR} = (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{QO} + \overrightarrow{OR}) \\
= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{OR} - \overrightarrow{PO}) \\
= \|\overrightarrow{OR}\|^2 - \|\overrightarrow{PO}\|^2 \\
= r^2 - r^2 = 0,
\]

where \( r \) is the radius of the circle. It follows that \( \overrightarrow{PR} \) and \( \overrightarrow{QR} \) are indeed orthogonal. \( \square \)