2. Dot product

Definition 2.1. *Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be two vectors in $\mathbb{R}^3$. The* ***dot product*** *of $\vec{v}$ and $\vec{w}$, denoted $\vec{v} \cdot \vec{w}$, is the scalar $v_1w_1 + v_2w_2 + v_3w_3$.*

Example 2.2. *The dot product of $\vec{v} = (1, -2, -1)$ and $\vec{w} = (2, 1, -3)$ is*

$$1 \cdot 2 + (-2) \cdot 1 + (-1) \cdot (-3) = 2 - 2 + 3 = 3.$$

Lemma 2.3. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in $\mathbb{R}^3$ and let λ be a scalar.*

(1) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
(2) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
(3) $(\lambda\vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w})$.
(4) $\vec{v} \cdot \vec{v} = 0$ *if and only if* $\vec{v} = \vec{0}$.

Proof. (1–3) are straightforward.

To see (4), first note that one direction is clear. If $\vec{v} = \vec{0}$, then $\vec{v} \cdot \vec{v} = 0$. For the other direction, suppose that $\vec{v} \cdot \vec{v} = 0$. Then $v_1^2 + v_2^2 + v_3^2 = 0$. Now the square of a real number is non-negative and if a sum of non-negative numbers is zero, then each term must be zero. It follows that $v_1 = v_2 = v_3 = 0$ and so $\vec{v} = \vec{0}$. □

Definition 2.4. *If $\vec{v} \in \mathbb{R}^3$, then the* ***norm*** *or* ***length*** *of $\vec{v} = (v_1, v_2, v_3)$ is the scalar*

$$\|v\| = \sqrt{\vec{v} \cdot \vec{v}} = (v_1^2 + v_2^2 + v_3^2)^{1/2}.$$

It is interesting to note that if you know the norm, then you can calculate the dot product:

$$\begin{aligned}(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) &= \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \\ (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) &= \vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}.\end{aligned}$$

Subtracting and dividing by 4 we get

$$\begin{aligned}\vec{v} \cdot \vec{w} &= \frac{1}{4}\left((\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) - (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})\right) \\ &= \frac{1}{4}(\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2).\end{aligned}$$

Given two non-zero vectors $\vec{v}$ and $\vec{w}$ in space, note that we can define the angle θ between $\vec{v}$ and $\vec{w}$. $\vec{v}$ and $\vec{w}$ lie in at least one plane (which is in fact unique, unless $\vec{v}$ and $\vec{w}$ are parallel). Now just measure the angle θ between the $\vec{v}$ and $\vec{w}$ in this plane. By convention we always take $0 \leq \theta \leq \pi$.

Theorem 2.5. *If $\vec{v}$ and $\vec{w}$ are any two vectors in $\mathbb{R}^3$, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \; \|\vec{w}\| \cos\theta.$$

Proof. If $\vec{v}$ is the zero vector, then both sides are equal to zero, so that they are equal to each other and the formula holds (note though, that in this case the angle θ is not determined).

By symmetry, we may assume that $\vec{v}$ and $\vec{w}$ are both non-zero. Let $\vec{u} = \vec{w} - \vec{v}$ and apply the law of cosines to the triangle with sides parallel to $\vec{u}$, $\vec{v}$ and $\vec{w}$:

$$\|\vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta.$$

We have already seen that the LHS of this equation expands to

$$\vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = \|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|.$$

Cancelling the common terms $\|\vec{v}\|^2$ and $\|\vec{w}\|^2$ from both sides, and dividing by 2, we get the desired formula. □

We can use (2.5) to find the angle between two vectors:

Example 2.6. *Let $\vec{v} = -\hat{\imath} + \hat{k}$ and $\vec{w} = \hat{\imath} + \hat{\jmath}$. Then*

$$-1 = \vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = 2\cos\theta.$$

Therefore $\cos\theta = -1/2$ and so $\theta = 2\pi/3$.

Definition 2.7. *We say that two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.*

Remark 2.8. *If neither $\vec{v}$ nor $\vec{w}$ are the zero vector, and $\vec{v} \cdot \vec{w} = 0$ then the angle between $\vec{v}$ and $\vec{w}$ is a right angle. Our convention is that the zero vector is orthogonal to every vector.*

Example 2.9. *$\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are pairwise orthogonal.*

Given two vectors $\vec{v}$ and $\vec{w}$, we can project $\vec{v}$ onto $\vec{w}$. The resulting vector is called the **projection** of $\vec{v}$ onto $\vec{w}$ and is denoted $\operatorname{proj}_{\vec{w}} \vec{v}$. For example, if $\vec{F}$ is a force and $\vec{w}$ is a direction, then the projection of $\vec{F}$ onto $\vec{w}$ is the force in the direction of $\vec{w}$.

As $\operatorname{proj}_{\vec{w}} \vec{v}$ is parallel to $\vec{w}$, we have

$$\operatorname{proj}_{\vec{w}} \vec{v} = \lambda \vec{w},$$

for some scalar λ. Let's determine λ. Let's deal with the case that $\lambda \geq 0$ (so that the angle θ between $\vec{v}$ and $\vec{w}$ is between 0 and $\pi/2$). If we take the norm of both sides, we get

$$\|\operatorname{proj}_{\vec{w}} \vec{v}\| = \|\lambda \vec{w}\| = \lambda \|\vec{w}\|,$$

(note that $\lambda \geq 0$), so that

$$\lambda = \frac{\| \operatorname{proj}_{\vec{w}} \vec{v}\|}{\|\vec{w}\|}.$$

But

$$\cos\theta = \frac{\| \operatorname{proj}_{\vec{w}} \vec{v}\|}{\|\vec{v}\|},$$

so that

$$\| \operatorname{proj}_{\vec{w}} \vec{v}\| = \|\vec{v}\| \cos\theta.$$

Putting all of this together we get

$$\begin{aligned}\lambda &= \frac{\|\vec{v}\| \cos\theta}{\|\vec{w}\|} \\ &= \frac{\|\vec{v}\|\|\vec{w}\| \cos\theta}{\|\vec{w}\|^2} \\ &= \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}.\end{aligned}$$

There are a number of ways to deal with the case when $\lambda < 0$ (so that $\theta > \pi/2$). One can carry out a similar analysis to the one given above. Here is another way. Note that the angle ϕ between $\vec{w}$ and $\vec{u} = -\vec{v}$ is equal to $\pi - \theta < \pi/2$. By what we already proved

$$\operatorname{proj}_{\vec{w}} \vec{u} = \frac{\vec{u} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}.$$

But $\operatorname{proj}_{\vec{w}} \vec{u} = -\operatorname{proj}_{\vec{w}} \vec{v}$ and $\vec{u} \cdot \vec{w} = -\vec{v} \cdot \vec{w}$, so we get the same formula in the end. To summarise:

Theorem 2.10. *If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^3$, where $\vec{w}$ is not zero, then*

$$\operatorname{proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right) \vec{w}.$$

Example 2.11. *Find the distance d between the line l containing the points $P_1 = (1, -1, 2)$ and $P_2 = (4, 1, 0)$ and the point $Q = (3, 2, 4)$.*

Suppose that R is the closest point on the line l to the point Q. Note that $\overrightarrow{QR}$ is orthogonal to the direction $\overrightarrow{P_1P_2}$ of the line. So we want the length of the vector $\overrightarrow{P_1Q} - \operatorname{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q}$, that is, we want

$$d = \|\overrightarrow{P_1Q} - \operatorname{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q}\|.$$

Now

$$\overrightarrow{P_1Q} = (2, 3, 2) \qquad \text{and} \qquad \overrightarrow{P_1P_2} = (3, 2, -2).$$

We have

$$\|\overrightarrow{P_1P_2}\|^2 = 3^2 + 2^2 + 2^2 = 17 \qquad \textit{and} \qquad \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q} = 6 + 6 - 4 = 8.$$

It follows that

$$\operatorname{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q} = \frac{8}{17}(3, 2, -2).$$

Subtracting, we get

$$\overrightarrow{P_1Q} - \operatorname{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q} = (2, 3, 2) - \frac{8}{17}(3, 2, -2) = \frac{1}{17}(10, 35, 50) = \frac{5}{17}(2, 7, 10).$$

Taking the length, we get

$$\frac{5}{17}(2^2 + 7^2 + 10^2)^{1/2} \approx 3.64.$$

Theorem 2.12. *The angle subtended on the circumference of a circle by a diameter of the circle is always a right angle.*

Proof. Suppose that P and Q are the two endpoints of a diameter of the circle and that R is a point on the circumference. We want to show that the angle between $\overrightarrow{PR}$ and $\overrightarrow{QR}$ is a right angle.

Let O be the centre of the circle. Then

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} \qquad \text{and} \qquad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}.$$

Note that $\overrightarrow{QO} = -\overrightarrow{PO}$. Therefore

$$\begin{aligned}
\overrightarrow{PR} \cdot \overrightarrow{QR} &= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{QO} + \overrightarrow{OR}) \\
&= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{OR} - \overrightarrow{PO}) \\
&= \|\overrightarrow{OR}\|^2 - \|\overrightarrow{PO}\|^2 \\
&= r^2 - r^2 = 0,
\end{aligned}$$

where r is the radius of the circle. It follows that $\overrightarrow{PR}$ and $\overrightarrow{QR}$ are indeed orthogonal. $\square$