17. Vector fields

Definition 17.1. Let $A \subset \mathbb{R}^n$ be an open subset. A vector field on $A$ is function $\vec{F} : A \to \mathbb{R}^n$.

One obvious way to get a vector field is to take the gradient of a differentiable function. If $f : A \to \mathbb{R}$, then

$$\nabla f : A \to \mathbb{R}^n,$$

is a vector field.

Definition 17.2. A vector field $\vec{F} : A \to \mathbb{R}^n$ is called a gradient (aka conservative) vector field if $\vec{F} = \nabla f$ for some differentiable function $f : A \to \mathbb{R}$.

Example 17.3. Let

$$\vec{F} : \mathbb{R}^3 - \{0\} \to \mathbb{R}^3,$$

be the vector field

$$\vec{F}(x, y, z) = \frac{cx}{(x^2 + y^2 + z^2)^{3/2}} \hat{i} + \frac{cy}{(x^2 + y^2 + z^2)^{3/2}} \hat{j} + \frac{cz}{(x^2 + y^2 + z^2)^{3/2}} \hat{k},$$

for some constant $c$. Then $\vec{F}(x, y, z)$ is the gradient of

$$f : \mathbb{R}^3 - \{0\} \to \mathbb{R},$$

given by

$$f(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}}.$$

So $\vec{F}$ is a conservative vector field. Notice that if $c < 0$ then $\vec{F}$ models the gravitational force and $f$ is the potential (note that unfortunately mathematicians and physicists have different sign conventions for $f$).

Proposition 17.4. If $\vec{F}$ is a conservative vector field and $\vec{F}$ is $C^1$ function, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

for all $i$ and $j$ between 1 and $n$.

Proof. If $\vec{F}$ is conservative, then we may find a differentiable function $f : A \to \mathbb{R}^n$ such that

$$F_i = \frac{\partial f}{\partial x_i}.$$
As $F_i$ is $C^1$ for each $i$, it follows that $f$ is $C^2$. But then

\[
\frac{\partial F_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial F_j}{\partial x_i}.
\]

Notice that (17.4) is a negative result; one can use it show that various vector fields are not conservative.

**Example 17.5.** Let

\[
\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (-y, x).
\]

Then

\[
\frac{\partial F_1}{\partial y} = -1 \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1 \neq -1.
\]

So $\vec{F}$ is not conservative.

**Example 17.6.** Let

\[
\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (y, x + y).
\]

Then

\[
\frac{\partial F_1}{\partial y} = 1 \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1,
\]

so $\vec{F}$ might be conservative. Let’s try to find

\[
f : \mathbb{R}^2 \rightarrow \mathbb{R} \quad \text{such that} \quad \nabla f(x, y) = (y, x + y).
\]

If $f$ exists, then we must have

\[
\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x + y.
\]

If we integrate the first equation with respect to $x$, then we get

\[
f(x, y) = xy + g(y).
\]

Note that $g(y)$ is not just a constant but it is a function of $y$. There are two ways to see this. One way, is to imagine that for every value of $y$, we have a separate differential equation. If we integrate both sides, we get an arbitrary constant $c$. As we vary $y$, $c$ varies, so that $c = g(y)$ is a function of $y$. On the other hand, if to take the partial derivatives
of \( g(y) \) with respect to \( x \), then we get 0. Now we take \( xy + g(y) \) and differentiate with respect to \( y \), to get
\[
x + y = \frac{\partial(xy + g(y))}{\partial y} = x + \frac{dg}{dy}(y).
\]
So
\[
g'(y) = y.
\]
Integrating both sides with respect to \( y \) we get
\[
g(y) = \frac{y^2}{2} + c.
\]
It follows that
\[
\nabla(xy + \frac{y^2}{2}) = (y, x + y),
\]
so that \( \vec{F} \) is conservative.

**Definition 17.7.** If \( \vec{F}: A \rightarrow \mathbb{R}^n \) is a vector field, we say that a parametrised differentiable curve \( \vec{r}: I \rightarrow A \) is a **flow line** for \( \vec{F} \), if
\[
\vec{r}'(t) = \vec{F}(\vec{r}(t)),
\]
for all \( t \in I \).

**Example 17.8.** Let
\[
\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (-y, x).
\]
We check that
\[
\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{r}(t) = (a \cos t, a \sin t),
\]
is a flow line. In fact
\[
\vec{r}'(t) = (-a \sin t, a \cos t),
\]
and so
\[
\vec{F}(\vec{r}(t)) = \vec{F}(a \cos t, a \sin t)
\]
\[
= \vec{r}'(t),
\]
so that \( \vec{r}(t) \) is indeed a flow line.

**Example 17.9.** Let
\[
\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad \vec{F}(x, y) = (-x, y).
\]
Let’s find a flow line through the point \((a, b)\). We have
\[
x'(t) = -x(t) \quad x(0) = a
\]
\[
y'(t) = y(t) \quad y(0) = b.
\]
Therefore,
\[
x(t) = ae^{-t} \quad \text{and} \quad y(t) = be^t,
\]
gives the flow line through \((a, b)\).
Example 17.10. Let 
\[ \vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \] 
given by 
\[ \vec{F}(x, y) = (x^2 - y^2, 2xy). \]

Try 
\[ x(t) = 2a \cos t \sin t \]
\[ y(t) = 2a \sin^2 t. \]

Then 
\[ x'(t) = 2a(- \sin^2 t + \cos^t) \]
\[ = \frac{x^2(t) - y^2(t)}{y(t)}. \]

Similarly 
\[ y'(t) = 4a \cos t \sin t \]
\[ = \frac{2x(t)y(t)}{y(t)}. \]

So 
\[ \vec{r}'(t) = \frac{\vec{F}(\vec{r}(t))}{f(t)}. \]

So the curves themselves are flow lines, but this is not the correct parametrisation. The flow lines are circles passing through the origin, with centre along the y-axis.

Example 17.11. Let 
\[ \vec{F} : \mathbb{R}^2 - \{(0, 0)\} \to \mathbb{R}^2 \] 
given by 
\[ \vec{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right). \]

Then 
\[ \frac{\partial F_1}{\partial y}(x, y) = - \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \]

and 
\[ \frac{\partial F_2}{\partial x}(x, y) = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \]

So \( \vec{F} \) might be conservative. Let’s find the flow lines. Try 
\[ x(t) = a \cos \left(\frac{t}{a^2}\right) \]
\[ y(t) = a \sin \left(\frac{t}{a^2}\right). \]
Then

\[ x'(t) = -\frac{1}{a} \sin \left( \frac{t}{a^2} \right) \]
\[ = -\frac{y}{x^2 + y^2}. \]

Similarly

\[ y'(t) = \frac{1}{a} \cos \left( \frac{t}{a^2} \right) \]
\[ = \frac{x}{x^2 + y^2}. \]

So the flow lines are closed curves. In fact this means that \( \vec{F} \) is not conservative.