17. Vector fields

Definition 17.1. *Let $A \subset \mathbb{R}^n$ be an open subset. A* ***vector field*** *on A is function $\vec{F}\colon A \longrightarrow \mathbb{R}^n$.*

One obvious way to get a vector field is to take the gradient of a differentiable function. If $f\colon A \longrightarrow \mathbb{R}$, then

$$\nabla f\colon A \longrightarrow \mathbb{R}^n,$$

is a vector field.

Definition 17.2. *A vector field $\vec{F}\colon A \longrightarrow \mathbb{R}^n$ is called a* ***gradient (aka conservative) vector field*** *if $\vec{F} = \nabla f$ for some differentiable function $f\colon A \longrightarrow \mathbb{R}$.*

Example 17.3. *Let*

$$\vec{F}\colon \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}^3,$$

be the vector field

$$\vec{F}(x,y,z) = \frac{cx}{(x^2+y^2+z^2)^{3/2}}\hat{\imath} + \frac{cy}{(x^2+y^2+z^2)^{3/2}}\hat{\jmath} + \frac{cz}{(x^2+y^2+z^2)^{3/2}}\hat{k},$$

for some constant c. Then $\vec{F}(x,y,z)$ is the gradient of

$$f\colon \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R},$$

given by

$$f(x,y,z) = -\frac{c}{(x^2+y^2+z^2)^{1/2}}.$$

So $\vec{F}$ is a conservative vector field. Notice that if $c < 0$ then $\vec{F}$ models the gravitational force and f is the potential (note that unfortunately mathematicians and physicists have different sign conventions for f).

Proposition 17.4. *If $\vec{F}$ is a conservative vector field and $\vec{F}$ is $\mathcal{C}^1$ function, then*

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i},$$

for all i and j between 1 and n.

Proof. If $\vec{F}$ is conservative, then we may find a differentiable function $f\colon A \longrightarrow \mathbb{R}^n$ such that

$$F_i = \frac{\partial f}{\partial x_i}.$$

As F_i is $\mathcal{C}^1$ for each i, it follows that f is $\mathcal{C}^2$. But then

$$\begin{aligned}\frac{\partial F_i}{\partial x_j} &= \frac{\partial^2 f}{\partial x_j \partial x_i}\\ &= \frac{\partial^2 f}{\partial x_i \partial x_j}\\ &= \frac{\partial F_j}{\partial x_i}.\end{aligned}$$ □

Notice that (17.4) is a negative result; one can use it show that various vector fields are not conservative.

Example 17.5. *Let*

$$\vec{F} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x,y) = (-y, x).$$

Then

$$\frac{\partial F_1}{\partial y} = -1 \qquad \text{and} \qquad \frac{\partial F_2}{\partial x} = 1 \neq -1.$$

So $\vec{F}$ is not conservative.

Example 17.6. *Let*

$$\vec{F} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x,y) = (y, x+y).$$

Then

$$\frac{\partial F_1}{\partial y} = 1 \qquad \text{and} \qquad \frac{\partial F_2}{\partial x} = 1,$$

so $\vec{F}$ might be conservative. Let's try to find

$$f \colon \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \text{such that} \qquad \nabla f(x,y) = (y, x+y).$$

If f exists, then we must have

$$\frac{\partial f}{\partial x} = y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x+y.$$

If we integrate the first equation with respect to x, then we get

$$f(x,y) = xy + g(y).$$

Note that $g(y)$ is not just a constant but it is a function of y. There are two ways to see this. One way, is to imagine that for every value of y, we have a separate differential equation. If we integrate both sides, we get an arbitrary constant c. As we vary y, c varies, so that $c = g(y)$ is a function of y. On the other hand, if to take the partial derivatives *of $g(y)$ with respect to x, then we get 0. Now we take $xy + g(y)$ and differentiate with respect to y, to get*

$$x + y = \frac{\partial(xy + g(y))}{\partial y} = x + \frac{dg}{dy}(y).$$

So

$$g'(y) = y.$$

Integrating both sides with respect to y we get

$$g(y) = y^2/2 + c.$$

It follows that

$$\nabla(xy + y^2/2) = (y, x + y),$$

so that $\vec{F}$ is conservative.

Definition 17.7. *If $\vec{F}\colon A \longrightarrow \mathbb{R}^n$ is a vector field, we say that a parametrised differentiable curve $\vec{r}\colon I \longrightarrow A$ is a* ***flow line*** *for $\vec{F}$, if*

$$\vec{r}'(t) = \vec{F}(\vec{r}(t)),$$

for all $t \in I$.

Example 17.8. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x, y) = (-y, x).$$

We check that

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{r}(t) = (a\cos t, a\sin t),$$

is a flow line. In fact

$$\vec{r}'(t) = (-a\sin t, a\cos t),$$

and so

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= \vec{F}(a\cos t, a\sin t)\\ &= \vec{r}'(t),\end{aligned}$$

so that $\vec{r}(t)$ is indeed a flow line.

Example 17.9. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x, y) = (-x, y).$$

Let's find a flow line through the point (a, b). We have

$$\begin{aligned} x'(t) &= -x(t) & \qquad x(0) &= a\\ y'(t) &= y(t) & \qquad y(0) &= b.\end{aligned}$$

Therefore,

$$x(t) = ae^{-t} \qquad \text{and} \qquad y(t) = be^{t},$$

gives the flow line through (a, b).

Example 17.10. *Let*

$$\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x,y) = (x^2 - y^2, 2xy).$$

Try

$$\begin{aligned} x(t) &= 2a\cos t \sin t \\ y(t) &= 2a\sin^2 t. \end{aligned}$$

Then

$$\begin{aligned} x'(t) &= 2a(-\sin^2 t + \cos^t) \\ &= \frac{x^2(t) - y^2(t)}{y(t)}. \end{aligned}$$

Similarly

$$\begin{aligned} y'(t) &= 4a\cos t \sin t \\ &= \frac{2x(t)y(t)}{y(t)}. \end{aligned}$$

So

$$\vec{r}'(t) = \frac{\vec{F}(\vec{r}(t))}{f(t)}.$$

So the curves themselves are flow lines, but this is not the correct parametrisation. The flow lines are circles passing through the origin, with centre along the y-axis.

Example 17.11. *Let*

$$\vec{F}\colon \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad \vec{F}(x,y) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}).$$

Then

$$\frac{\partial F_1}{\partial y}(x,y) = -\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

and

$$\frac{\partial F_2}{\partial x}(x,y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

So $\vec{F}$ might be conservative. Let's find the flow lines. Try

$$\begin{aligned} x(t) &= a\cos\left(\frac{t}{a^2}\right) \\ y(t) &= a\sin\left(\frac{t}{a^2}\right). \end{aligned}$$

Then

$$x'(t) = -\frac{1}{a}\sin\left(\frac{t}{a^2}\right)$$
$$= -\frac{y}{x^2+y^2}.$$

Similarly

$$y'(t) = \frac{1}{a}\cos\left(\frac{t}{a^2}\right)$$
$$= \frac{x}{x^2+y^2}.$$

So the flow lines are closed curves. In fact this means that $\vec{F}$ *is not conservative.*