

16. MOVING FRAMES

Definition 16.1. We say a parametrised differentiable curve $\vec{r}: I \longrightarrow \mathbb{R}^n$ is **regular** if $\vec{r}'(t) \neq 0$ (the speed is never zero).

We say that $\vec{r}(t)$ is **smooth** if $\vec{r}(t)$ is C^∞ .

Given a regular smooth parametrised differentiable curve $\vec{r}: I \longrightarrow \mathbb{R}^3$, we can parametrise by arclength, in which case we can write down the unit tangent vector

$$\vec{T} = \frac{d\vec{r}}{ds}(s).$$

The curvature $\kappa(s)$ is defined as the magnitude of

$$\frac{d\vec{T}}{ds}(s).$$

If the curvature is nowhere zero, then we define the normal vector $\vec{N}(s)$ as the unit vector pointing in the direction of the derivative of the tangent vector:

$$\frac{d\vec{T}}{ds}(s) = \kappa(s)\vec{N}(s).$$

We have already seen that $\vec{T}(s)$ and $\vec{N}(s)$ are orthogonal.

Definition 16.2.

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s).$$

is called the **binormal vector**.

The three vectors $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ are unit vectors and pairwise orthogonal, that is, these vectors are an orthonormal basis of \mathbb{R}^3 . Notice that $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ are a right handed set.

We call these vectors a **moving frame** or the **Frenet-Serret frame**.
Now

$$\frac{d\vec{B}}{ds}(s) \times \vec{B}(s) = 0,$$

as

$$\|\vec{B}(s) \cdot \vec{B}(s)\| = 1.$$

It follows that

$$\frac{d\vec{B}}{ds}(s),$$

lies in the plane spanned by $\vec{T}(s)$ and $\vec{N}(s)$.

$$\begin{aligned}
\frac{d\vec{B}}{ds}(s) \cdot \vec{T}(s) &= \frac{d(\vec{T} \times \vec{N})}{ds}(s) \cdot \vec{T}(s) \\
&= \left(\frac{d\vec{T}}{ds}(s) \times \vec{N}(s) + \vec{T}(s) \times \frac{d\vec{N}}{ds}(s) \right) \cdot \vec{T}(s) \\
&= \kappa(s)(\vec{N}(s) \times \vec{N}(s)) \cdot \vec{T}(s) + (\vec{T}(s) \times \vec{N}(s)) \cdot \vec{T}(s) \\
&= 0 + (\vec{T}(s) \times \vec{T}(s)) \cdot \vec{N}(s) \\
&= 0.
\end{aligned}$$

It follows that

$$\frac{d\vec{B}}{ds}(s) \quad \text{and} \quad \vec{T}(s),$$

are orthogonal, and so

$$\frac{d\vec{B}}{ds}(s) \quad \text{is parallel to} \quad \vec{N}(s).$$

Definition 16.3. The *torsion* of the curve $\vec{r}(s)$ is the unique scalar $\tau(s)$ such that

$$\frac{d\vec{B}}{ds}(s) = -\tau(s)\vec{N}(s).$$

If we have a helix, the sign of the torsion distinguishes between a right handed helix and a left handed helix. The magnitude of the torsion measures how spread out the helix is (the curvature measures how tight the turns are). Now

$$\frac{d\vec{N}}{ds}(s)$$

is orthogonal to $\vec{N}(s)$, and so it is a linear combination of $\vec{T}(s)$ and $\vec{B}(s)$. In fact,

$$\begin{aligned}
\frac{d\vec{N}}{ds}(s) &= \frac{d(\vec{B} \times \vec{T})}{ds}(s) \\
&= \frac{d\vec{B}}{ds}(s) \times \vec{T}(s) + \vec{B}(s) \times \frac{d\vec{T}}{ds}(s) \\
&= -\tau(s)\vec{N}(s) \times \vec{T}(s) + \kappa(s)\vec{B}(s) \times \vec{N}(s) \\
&= \tau(s)\vec{B}(s) - \kappa(s)\vec{T}(s) \\
&= -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s).
\end{aligned}$$

Theorem 16.4 (Frenet Formulae). *Let $\vec{r}: I \rightarrow \mathbb{R}^3$ be a regular smooth parametrised curve. Then*

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}.$$

Of course, s represents the arclength parameter and primes denote derivatives with respect to s . Notice that the 3×3 matrix A appearing in (16.4) is **skew-symmetric**, that is $A^t = -A$. The way we have written the Frenet formulae, it appears that we have two 3×1 vectors; strictly speaking these are the rows of two 3×3 matrices.

Theorem 16.5. *Let $I \subset \mathbb{R}$ be an open interval and suppose we are given two smooth functions*

$$\kappa: I \rightarrow \mathbb{R} \quad \text{and} \quad \tau: I \rightarrow \mathbb{R},$$

where $\kappa(s) > 0$ for all $s \in I$.

Then there is a regular smooth curve $\vec{r}: I \rightarrow \mathbb{R}^3$ parametrised by arclength with curvature $\kappa(s)$ and torsion $\tau(s)$. Further, any two such curves are congruent, that is, they are the same up to translation and rotation.

Remark 16.6. *Uniqueness is one of the hwk problems.*

Let's consider the example of the helix:

Example 16.7.

$$\vec{r}(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c} \right),$$

where

$$c^2 = a^2 + b^2.$$

Let's assume that $a > 0$. By convention $c > 0$. Then

$$\vec{T}(s) = \frac{1}{c} \left(-a \sin \frac{s}{c}, a \cos \frac{s}{c}, b \right).$$

Hence

$$\frac{d\vec{T}}{ds}(s) = \frac{-a}{c^2} \left(\cos \frac{s}{c}, \sin \frac{s}{c}, 0 \right) = \frac{a}{c^2} \left(-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0 \right) = \frac{a}{c^2} \vec{N}(s)$$

It follows that

$$\kappa(s) = \frac{a}{c^2} \quad \text{and} \quad \vec{N}(s) = \left(-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0 \right).$$

Finally,

$$\vec{B}(s) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{a}{c} \sin \frac{s}{c} & \frac{a}{c} \cos \frac{s}{c} & \frac{b}{c} \\ -\cos \frac{s}{c} & -\sin \frac{s}{c} & 0 \end{vmatrix}$$

It follows that

$$\vec{B}(s) = \left(\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c}\right) = \frac{1}{c}(b \sin \frac{s}{c}, -b \cos \frac{s}{c}, a).$$

Finally, note that

$$\frac{d\vec{B}}{ds}(s) = \frac{b}{c^2}(\cos \frac{s}{c}, \sin \frac{s}{c}, 0) = -\frac{b}{c^2}\vec{N}.$$

Using this we can compute the torsion:

$$\tau(s) = \frac{b}{c^2}.$$

It is interesting to use the torsion and curvature to characterise various geometric properties of curves. Let's say that a parametrised differentiable curve $\vec{r}: I \rightarrow \mathbb{R}^3$ is **planar** if there is a plane Π which contains the image of \vec{r} .

Theorem 16.8. *A regular smooth curve $\vec{r}: I \rightarrow \mathbb{R}^3$ is planar if and only if the torsion is zero.*

Proof. We may assume that the curve passes through the origin.

Suppose that \vec{r} is planar. Then the image of \vec{r} is contained in a plane Π . As the curve passes through the origin, Π contains the origin as well. Note that the unit tangent vector $\vec{T}(s)$ and the unit normal vector $\vec{N}(s)$ are contained in Π . It follows that $\vec{B}(s)$ is a normal vector to the plane; as $\vec{B}(s)$ is a unit vector, it must be constant. But then

$$\frac{d\vec{B}}{ds}(s) = \vec{0} = 0\vec{N}(s),$$

so that the torsion is zero.

Now suppose that the torsion is zero. Then

$$\frac{d\vec{B}}{ds}(s) = 0\vec{N} = \vec{0},$$

so that $\vec{B}(s) = \vec{B}_0$, is a constant vector. Consider the function

$$f(s) = \vec{r}(s) \cdot \vec{B}(s) = \vec{r}(s) \cdot \vec{B}_0.$$

Then

$$\begin{aligned} \frac{df}{ds}(s) &= \frac{d(\vec{r} \cdot \vec{B}_0)}{ds}(s) \\ &= \vec{T}(s) \cdot \vec{B}_0 = 0. \end{aligned}$$

So $f(s)$ is constant. It is zero when $\vec{r}(a) = \vec{0}$ (the curve passes through the origin) so that $f(s) = 0$. But then $\vec{r}(s)$ is always orthogonal to a fixed vector, so that \vec{r} is contained in a plane, that is, C is planar. \square

It is interesting to try to figure out how to characterise curves which are contained in spheres or cylinders.