Definition 15.1. *Let I be an open interval. A **partition** $\mathcal{P}$ of $[a, b] \subset I$ is a sequence of points*

$$a = t_0 < t_1 < \cdots < t_n = b,$$

*for some $n \geq 1$. The **mesh** of $\mathcal{P}$ is*

$$m(\mathcal{P}) = \max_i \{ \, |t_{i+1} - t_i| \, | \, 0 \leq i \leq n - 1 \, \}.$$

Definition 15.2. *Let $\vec{r} \colon I \longrightarrow \mathbb{R}^n$ be a parametrised differentiable curve, and let $\mathcal{P}$ be a partition. We think of*

$$l(\mathcal{P}) = \sum_{i=0}^{n-1} \| \vec{r}(t_{i+1}) - r(t_i) \|,$$

as being an approximation to the length of the curve $\vec{r}([a, b])$.

*The curve $\vec{r}$ has **length** L if given any $\epsilon > 0$, there is a $\delta > 0$ such that for every partition $\mathcal{P}$ whose mesh size is less than δ, we have*

$$\| L - l(\mathcal{P}) \| < \epsilon.$$

There are interesting examples of curves that don't have a length.

Start with interval $[0, 1]$ in the plane. The length is 1. Now adjust this curve, by adding in a triangular hump in the middle, so that we get four line segments of length $1/3$. This is a curve of length $4/3$.

Now add a hump to each of the four line segments of length $1/3$. The length of the resulting curve is $(4/3)^2$. If we keep doing this, then we get more and more complicated curves, whose length at stage n is $(4/3)^n$. This process converges to a very pointed curve whose length is infinite (in fact this curve is a fractal).

However:

Proposition 15.3. *If $\vec{r} \colon I \longrightarrow \mathbb{R}^n$ is a $\mathcal{C}^1$-function, then the curve $\vec{r}([a, b])$ has a length*

$$L = \int_a^b \| \vec{r}'(t) \| \, \mathrm{d}t.$$

Remark 15.4. *The fractal curve above is continuous but it is nowhere differentiable (the curve has too many sharp points).*

In general the exact formula for the arclength is only of theoretical interest. However there are some contrived examples where we can calculately the arclength precisely.

Example 15.5. *Let*

$$\vec{r} \colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

be the parametrised differentiable curve given by

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath}.$$

Then

$$\vec{r}'(t) = -a\sin t\hat{\imath} + a\cos t\hat{\jmath},$$

and so

$$\|\vec{r}'(t)\| = a.$$

Hence the length of the curve $\vec{r}([0, 2\pi])$ is

$$L = \int_0^{2\pi} a\, dt = 2\pi a,$$

which is indeed the circumference of a circle of radius a.

Example 15.6. *Let*

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^3,$$

be the parametrised differentiable curve given by

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath} + bt\hat{k},$$

Then

$$\vec{r}'(t) = -a\sin t\hat{\imath} + a\cos t\hat{\jmath} + b\hat{k},$$

and so

$$\|\vec{r}'(t)\| = \sqrt{a^2 + b^2}.$$

Hence the length of the curve $\vec{r}([0, 2\pi])$ is

$$L = \int_0^{2\pi} (a^2 + b^2)^{1/2}\, dt = 2\pi(a^2 + b^2)^{1/2}.$$

Example 15.7. *Let*

$$\vec{r}\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

be the parametrised differentiable curve given by

$$\vec{r}(t) = a\cos t\hat{\imath} + b\sin t\hat{\jmath}.$$

Then

$$\vec{r}'(t) = -a\sin t\hat{\imath} + b\cos t\hat{\jmath},$$

and so

$$\|\vec{r}'(t)\| = (a^2\sin^2 t + b^2\cos^2 t)^{1/2}$$

Hence the length of the curve $\vec{r}([0, 2\pi])$ is

$$L = \int_0^{2\pi} (a^2\sin^2 t + b^2\cos^2 t)^{1/2}\, dt,$$

the length of an ellipse, with major and minor axes of length a and b.

Definition 15.8. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a parametrised differentiable curve, which is of class $\mathcal{C}^1$. Suppose that $\vec{r}'(t)$ is nowhere zero. Given $a \in I$, define the **arclength parameter** $s(t)$, by the formula*

$$s(t) = \int_a^t \|\vec{r}'(\tau)\|\,\mathrm{d}\tau = \begin{cases} \text{length of } r([a, t]) & \text{if } t \geq a \\ \text{length of } -r([t, a]) & \text{if } t < a. \end{cases}$$

By the fundamental theorem of calculus

$$s'(t) = \frac{d}{dt} \int_a^t \|\vec{r}'(\tau)\|\,\mathrm{d}\tau$$
$$= \|\vec{r}'(t)\|,$$

which is the speed at time t. Since $s'(t)$ is nowhere zero, we can write t as a function of s, that is we can write down the inverse function, $t(s)$, which will be $\mathcal{C}^1$.

Example 15.9. *For the helix,*

$$\vec{r}(t) = a \cos t \hat{\imath} + a \sin t \hat{\jmath} + b\hat{k}t,$$

we have

$$s(t) = \int_0^t \|\vec{r}'(\tau)\|\,\mathrm{d}\tau = \sqrt{a^2 + b^2}^{1/2} t.$$

Therefore

$$t(s) = \frac{s}{\sqrt{a^2 + b^2}}.$$

In this case

$$\vec{r}(s) = a \cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right) \hat{\imath} + a \sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right) \hat{\jmath} + b\hat{k}\left(\frac{s}{\sqrt{a^2 + b^2}}\right).$$

In fact, one can always parametrise a curve by its arclength and in this case the derivative is a unit vector:

Definition 15.10. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a parametrised differentiable curve parametrised by the arclength. Then*

$$\vec{T}(s) = \frac{d\vec{r}}{ds},$$

*is the **unit tangent vector**.*

In the case of the helix, we get

$$\vec{r}(s) = -\frac{a}{\sqrt{a^2 + b^2}} \sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right) \hat{\imath} + \frac{a}{\sqrt{a^2 + b^2}} \cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right) \hat{\jmath} + \left(\frac{b}{\sqrt{a^2 + b^2}}\right) \hat{k}.$$

Note that there is an obvious way to get the unit tangent vector. Take the derivative and divide by the speed. In fact

$$\frac{d\vec{r}(t)}{dt} = \frac{d\vec{r}(s)}{ds}\frac{ds}{dt}$$
$$= \vec{T}(s)\frac{ds}{dt},$$

by the chain rule. So

$$\vec{T}(s) = \frac{\frac{d\vec{r}(t)}{dt}}{\frac{ds}{dt}},$$

where the denominator is precisely the speed, so that the unit tangent vector is the unit vector which points in the direction of the velocity.

Example 15.11. *Let $\vec{r}(t) = (a\cos t, a\sin t)$ be the standard parametrisation of the circle of radius a. Then*

$$s = ta \qquad \text{so that} \qquad t = \frac{s}{a}.$$

So

$$\vec{r}(s) = (a\cos\left(\frac{s}{a}\right), a\sin\left(\frac{s}{a}\right)),$$

and

$$\vec{T}(s) = (-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right)).$$

Definition 15.12. *Let $\vec{r}\colon I \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^2$ curve such that $\vec{r}'(t) \neq \vec{0}$ for all t.*

*The **curvature** $\kappa(s)$ of $\vec{r}(s)$ is the magnitude of the vector*

$$\frac{d\vec{T}(s)}{ds},$$

*and the **unit normal vector** is the unit vector pointing in the direction of*

$$\vec{N}(s) = \frac{d\vec{T}(s)}{ds}.$$

In the example above,

$$\frac{d\vec{T}(s)}{ds} = \frac{1}{a}(-\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right)),$$

which has length $1/a$. So the larger the radius, the smaller the curvature (which is what one might expect). The unit normal vector is

$$\vec{N}(s) = (-\cos\left(\frac{s}{a}\right), -\sin\left(\frac{s}{a}\right)) = -\vec{r}(s).$$

One can try to calculate the curvature using the parameter t. By the chain rule,
$$\frac{d\vec{T}(t)}{dt} = \frac{d\vec{T}(s)}{ds}\frac{ds}{dt}.$$
So
$$\frac{d\vec{T}(s)}{ds} = \frac{\frac{d\vec{T}(t)}{dt}}{\frac{ds}{dt}}.$$
The denominator is the speed. It follows that
$$\vec{N}(s) \quad \text{and} \quad \frac{d\vec{T}(t)}{dt},$$
point in the same direction, so that the unit normal vector is the unit vector associated to
$$\frac{d\vec{T}(t)}{dt}.$$
On the other hand, the curvature is
$$\frac{\left\|\frac{d\vec{T}(t)}{dt}\right\|}{\frac{ds}{dt}}.$$

Note that the normal vector and the unit tangent vector are always orthogonal.

Indeed,
$$\|\vec{T}(s)\| = 1.$$
So
$$\vec{T}(s) \cdot \vec{T}(s) = 1.$$
Differentiate both sides with respect to s, to get
$$\vec{T}'(s) \cdot \vec{T}(s) + \vec{T}(s) \cdot \vec{T}'(s) = 0.$$
It follows that
$$2\vec{T}'(s) \cdot \vec{T}(s) = 0,$$
that is
$$\vec{T}'(s) \cdot \vec{T}(s) = 0.$$
As $\vec{N}(s)$ points in the same direction as $\vec{T}'(s)$, it follows that the tangent vector and the normal vector are orthogonal.