

14. PARAMETRISED CURVES

Definition 14.1. A *parametrised differentiable curve in \mathbb{R}^n* is a differentiable function $\vec{r}: I \rightarrow \mathbb{R}^n$, where I is an open interval in \mathbb{R} .

Remark 14.2. Any open interval I is one of four different forms: (a, b) ; $(-\infty, b)$; (a, ∞) ; $(-\infty, \infty) = \mathbb{R}$, where a and b are real numbers.

Definition 14.3. The *velocity vector* at time t of a parametrised differentiable curve $\vec{r}: I \rightarrow \mathbb{R}^n$ is the derivative:

$$\vec{v}(t) = \vec{r}'(t) = D\vec{r}(t).$$

If \vec{v} is differentiable, then the *acceleration vector* at time t is the derivative of the velocity vector:

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

Example 14.4. Let

$$\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3,$$

be given by

$$\vec{r}(t) = (a \cos t, a \sin t, bt).$$

This traces out a helix. The velocity vector is

$$\vec{v}(t) = (-a \sin t, a \cos t, b).$$

The acceleration vector is

$$\vec{a}(t) = (-a \cos t, -a \sin t, 0).$$

The speed, that is the magnitude of the velocity vector,

$$\|\vec{v}(t)\| = (a^2 + b^2)^{1/2},$$

is constant. Nevertheless the acceleration vector is not zero, as we are travelling on a curve and not a straight line.

Let's now attack a very famous problem. Kepler formulated three laws of planetary motion, based on extensive observations of the recorded positions of the planets. The first law states that planets move around in ellipses, where the sun is at one focal point of the ellipse; let's see how one can derive this law from Newton's universal law of gravity.

Let's put the sun at the origin O of our coordinates. Let's suppose that the planet is at the point $P = P(t)$ at time t . Then

$$\vec{r}(t): \mathbb{R} \rightarrow \mathbb{R}^3,$$

is a parametrised differentiable curve, where $\vec{r}(t) = \overrightarrow{OP}$.

We will need a simple formula for the vector triple product in \mathbb{R}^3 :

$$(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}.$$

One can check this formula using coordinates.

Theorem 14.5 (Newton). *Suppose that*

$$\vec{a} = -\frac{GM}{r^3}\vec{r},$$

for some constants G and M .

Then \vec{r} traces out either an ellipse, a parabola or a hyperbola.

Proof. We have

$$\begin{aligned}\frac{d(\vec{r} \times \vec{v})}{dt} &= \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} \\ &= \vec{v} \times \vec{v} + \vec{r} \times \vec{a} \\ &= \vec{0} + \vec{0} = \vec{0},\end{aligned}$$

since \vec{a} and \vec{r} are parallel by assumption. Hence

$$\vec{r} \times \vec{v} = \vec{c},$$

a constant vector. It follows that \vec{r} and \vec{v} lie in the plane Π through the origin and orthogonal to \vec{c} .

We may write $\vec{r} = r\hat{u}$, where \hat{u} is a unit vector.

$$\vec{v} = \frac{d(r\hat{u})}{dt} = \frac{dr}{dt}\hat{u} + r\frac{d\hat{u}}{dt}.$$

It follows that

$$\begin{aligned}\vec{c} &= \vec{r} \times \vec{v} \\ &= r\hat{u} \times \frac{dr}{dt}\hat{u} + r\hat{u} \times r\frac{d\hat{u}}{dt} \\ &= r^2\hat{u} \times \frac{d\hat{u}}{dt}.\end{aligned}$$

So

$$\begin{aligned}\vec{a} \times \vec{c} &= \left(-\frac{GM}{r^2}\hat{u}\right) \times \left(r^2\hat{u} \times \frac{d\hat{u}}{dt}\right) \\ &= -GM\hat{u} \times \left(\hat{u} \times \frac{d\hat{u}}{dt}\right) \\ &= GM\left(\hat{u} \times \frac{d\hat{u}}{dt}\right) \times \hat{u} \\ &= GM\left((\hat{u} \cdot \hat{u})\frac{d\hat{u}}{dt} - (\hat{u} \cdot \frac{d\hat{u}}{dt})\hat{u}\right) \\ &= \frac{d(GM\hat{u})}{dt}.\end{aligned}$$

On the other hand,

$$\vec{a} \times \vec{c} = \frac{d\vec{v}}{dt} \times \vec{c} = \frac{d(\vec{v} \times \vec{c})}{dt}$$

It follows that

$$\vec{v} \times \vec{c} = GM\hat{u} + \vec{d},$$

where \vec{d} is a constant vector. If we cross both sides with \vec{c} , then the LHS is zero and so the RHS is zero. It follows that d also lies in the plane Π . Define θ to be the angle between \vec{d} and \vec{u} . Now

$$\begin{aligned} \|\vec{c}\|^2 &= \vec{c} \cdot \vec{c} \\ &= (\vec{r} \times \vec{v}) \cdot \vec{c} \\ &= (\vec{v} \times \vec{c}) \cdot \vec{r} \\ &= (GM\hat{u} + \vec{d}) \cdot \vec{r} \\ &= GMr + r\|\vec{d}\| \cos \theta. \end{aligned}$$

Let $c = \|\vec{c}\|$ and $d = \|\vec{d}\|$. Then

$$\begin{aligned} r &= \frac{c^2}{GM + d \cos \theta} \\ &= \frac{p}{1 + e \cos \theta}, \end{aligned}$$

where

$$p = \frac{c^2}{GM} \quad \text{and} \quad e = \frac{d}{GM}.$$

Let's express these equations in Cartesian coordinates and not polar coordinates. We have

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta. \end{aligned}$$

Therefore

$$p = r + er \cos \theta,$$

so that

$$p = r + ex.$$

Solving for r ,

$$r = p - ex.$$

Squaring both sides we get

$$r^2 = (p - ex)^2.$$

That is

$$x^2 + y^2 = p^2 - 2epx + e^2x^2.$$

Therefore

$$(1 - e^2)x^2 + 2pex + y^2 = p^2.$$

There are three cases. The conic

$$C = \{ (x, y) \in \mathbb{R}^2 \mid (1 - e^2)x^2 + 2pex + y^2 = p^2 \},$$

is an

$$\begin{cases} \text{ellipse} & \text{if } |e| < 1 \\ \text{parabola} & \text{if } |e| = 1 \\ \text{hyperbola} & \text{if } |e| > 1. \end{cases}$$

Let's suppose that $|e| < 1$. First divide through by $1 - e^2$,

$$x^2 + \frac{2pe}{1 - e^2}x + \frac{1}{1 - e^2}y^2 = \frac{p^2}{1 - e^2}.$$

If we complete the square, then we get

$$\left(x + \frac{pe}{1 - e^2} \right)^2 + \frac{1}{1 - e^2}y^2 = \frac{p^2}{(1 - e^2)^2}.$$

Finally divide through by the RHS to get

$$\left(\frac{x + \frac{pe}{1 - e^2}}{\frac{p}{1 - e^2}} \right)^2 + \left(\frac{y}{\frac{p}{\sqrt{1 - e^2}}} \right)^2 = 1.$$

This is the equation of an ellipse. The centre of the ellipse is at

$$\left(-\frac{pe}{1 - e^2}, 0 \right).$$

One can check that this means one of the focal points is at the origin.

□