10. More about derivatives

The main result is:

Theorem 10.1. *Let $A \subset \mathbb{R}^n$ be an open subset and let $f\colon A \longrightarrow \mathbb{R}^m$ be a function.*

If the partial derivatives

$$\frac{\partial f_i}{\partial x_j},$$

exist and are continuous, then f is differentiable.

We will need:

Theorem 10.2 (Mean value theorem)**.** *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ is continuous and differentiable at every point of (a,b), then we may find $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

Geometrically, (10.2) is clear. However it is surprisingly hard to give a complete proof.

Proof of (10.1). We may assume that $m = 1$. We only prove this in the case when $n = 2$ (the general case is similar, only notationally more involved). So we have

$$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

Suppose that $P = (a,b)$ and let $\overrightarrow{PQ} = h_1\hat{\imath} + h_2\hat{\jmath}$. Let

$$P_0 = (a,b) \qquad P_1 = (a+h_1, b) \qquad \text{and} \qquad P_2 = (a+h_1, b+h_2) = Q.$$

Now

$$f(Q) - f(P) = [f(P_2) - f(P_1)] + [f(P_1) - f(P_0)].$$

We apply the Mean value theorem twice. We may find Q_1 and Q_2 such that

$$f(P_1) - f(P_0) = \frac{\partial f}{\partial x}(Q_1)h_1 \qquad \text{and} \qquad f(P_2) - f(P_1) = \frac{\partial f}{\partial y}(Q_2)h_2.$$

Here Q_1 lies somewhere on the line segment P_0P_1 and Q_2 lies on the line segment P_1P_2. Putting this together, we get

$$f(Q) - f(P) = \frac{\partial f}{\partial x}(Q_1)h_1 + \frac{\partial f}{\partial y}(Q_2)h_2.$$

Thus

$$\begin{aligned}
\frac{|f(Q)-f(P)-A\cdot\overrightarrow{PQ}|}{\|\overrightarrow{PQ}\|} &= \frac{|(\frac{\partial f}{\partial x}(Q_1)-\frac{\partial f}{\partial x}(P))h_1+(\frac{\partial f}{\partial y}(Q_2)-\frac{\partial f}{\partial y}(P))h_2|}{\|\overrightarrow{PQ}\|} \\
&\leq \frac{|(\frac{\partial f}{\partial x}(Q_1)-\frac{\partial f}{\partial x}(P))h_1|}{\|\overrightarrow{PQ}\|}+\frac{|(\frac{\partial f}{\partial y}(Q_2)-\frac{\partial f}{\partial y}(P))h_2|}{\|\overrightarrow{PQ}\|} \\
&\leq \frac{|(\frac{\partial f}{\partial x}(Q_1)-\frac{\partial f}{\partial x}(P))h_1|}{|h_1|}+\frac{|(\frac{\partial f}{\partial y}(Q_2)-\frac{\partial f}{\partial y}(P))h_2|}{|h_2|} \\
&= |(\frac{\partial f}{\partial x}(Q_1)-\frac{\partial f}{\partial x}(P))|+|(\frac{\partial f}{\partial y}(Q_2)-\frac{\partial f}{\partial y}(P))|.
\end{aligned}$$

Note that as Q approaches P, Q_1 and Q_2 both approach P as well. As the partials of f are continuous, we have

$$\lim_{Q\to P}\frac{|f(Q)-f(P)-A\cdot\overrightarrow{PQ}|}{\|\overrightarrow{PQ}\|}\leq \lim_{Q\to P}(|(\frac{\partial f}{\partial x}(Q_1)-\frac{\partial f}{\partial x}(P))|+|(\frac{\partial f}{\partial y}(Q_2)-\frac{\partial f}{\partial y}(P))|)=0.$$

Therefore f is differentiable at P, with derivative A. □

Example 10.3. *Let $f\colon A\longrightarrow\mathbb{R}$ be given by*

$$f(x,y)=\frac{x}{\sqrt{x^2+y^2}},$$

where $A=\mathbb{R}^2-\{(0,0)\}$. Then

$$\frac{\partial f}{\partial x}=\frac{(x^2+y^2)^{1/2}-x(2x)(1/2)(x^2+y^2)^{-1/2}}{x^2+y^2}=\frac{y^2}{(x^2+y^2)^{3/2}}.$$

Similarly

$$\frac{\partial f}{\partial y}=-\frac{xy}{(x^2+y^2)^{3/2}}.$$

Now both partial derivatives exist and are continuous, and so f is differentiable, with derivative the gradient,

$$\nabla f=(\frac{y^2}{(x^2+y^2)^{3/2}},-\frac{xy}{(x^2+y^2)^{3/2}})=\frac{1}{(x^2+y^2)^{3/2}}(y^2,-xy).$$

Lemma 10.4. *Let $A=(a_{ij})$ be an $m\times n$ matrix.*

If $\vec{v}\in\mathbb{R}^n$ then

$$\|A\vec{v}\|\leq K\|\vec{v}\|,$$

where

$$K=(\sum_{i,j}a_{ij}^2)^{1/2}.$$

Proof. Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ be the rows of A. Then the entry in the ith row of $A\vec{v}$ is $\vec{a}_i \cdot \vec{v}$. So,

$$\begin{aligned}\|A\vec{v}\|^2 &= (\vec{a}_1 \cdot \vec{v})^2 + (\vec{a}_2 \cdot \vec{v})^2 + \cdots + (\vec{a}_n \cdot \vec{v})^2 \\ &\leq \|\vec{a}_1\|^2\|\vec{v}\|^2 + \|\vec{a}_2\|^2\|\vec{v}\|^2 + \cdots + \|\vec{a}_n\|^2\|\vec{v}\|^2 \\ &= (\|\vec{a}_1\|^2 + \|\vec{a}_2\|^2 + \cdots + \|\vec{a}_n\|^2)\|\vec{v}\|^2 \\ &= K^2\|\vec{v}\|^2.\end{aligned}$$

Now take square roots of both sides. □

Theorem 10.5. *Let $f\colon A \longrightarrow \mathbb{R}^m$ be a function, where $A \subset \mathbb{R}^n$ is open.*

If f is differentiable at P, then f is continuous at P.

Proof. Suppose that $Df(P) = A$. Then

$$\lim_{Q \to P} \frac{f(Q) - f(P) - A \cdot \overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = 0.$$

This is the same as to require

$$\lim_{Q \to P} \frac{\|f(Q) - f(P) - A \cdot \overrightarrow{PQ}\|}{\|\overrightarrow{PQ}\|} = 0.$$

But if this happens, then surely

$$\lim_{Q \to P} \|f(Q) - f(P) - A \cdot \overrightarrow{PQ}\| = 0.$$

So

$$\begin{aligned}\|f(Q) - f(P)\| &= \|f(Q) - f(P) - A \cdot \overrightarrow{PQ} + A \cdot \overrightarrow{PQ}\| \\ &\leq \|f(Q) - f(P) - A \cdot \overrightarrow{PQ}\| + \|A \cdot \overrightarrow{PQ}\| \\ &\leq \|f(Q) - f(P) - A \cdot \overrightarrow{PQ}\| + K\|\overrightarrow{PQ}\|.\end{aligned}$$

Taking the limit as Q approaches P, both terms on the RHS go to zero, so that

$$\lim_{Q \to P} \|f(Q) - f(P)\| = 0,$$

and f is continuous at P. □