FIFTH HOMEWORK, DUE THURSDAY OCTOBER 14TH

Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

1. (10 pts) Find a recursive formula for a sequence of points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n), whose limit (x_∞, y_∞), if it exists, is a point of intersection of the curves

$$x^2 + y^2 = 1$$
$$x^2(x+1) = y^2.$$

2. (10 pts) Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(x,y) = \cos(xy) + x^3 + y^2$. Find a recursive formula for a sequence of points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n), whose limit (x_∞, y_∞), if it exists, is a **critical point** of the function f, that is, a point where $Df(x,y) = (0,0)$.

3. (10 pts) Suppose that $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is differentiable at $(-2,1)$ with derivative

$$Df(-2,1) = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix},$$

and that $f(-2,1) = (1,3)$. Let $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $g(y_1, y_2) = y_1^2 - y_2^2$.

(a) Show that the composite function $g \circ f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at $(-2,1)$.

(b) Find the derivative of $g \circ f$ at $(-2,1)$.

4. (10 pts) Let $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a $\mathcal{C}^1$ function. Suppose that $F(4,-1,2) = (0,0)$ and that

$$DF(4,-1,2) = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -1 \end{pmatrix}.$$

(a) Show that there is an open subset $U \subset \mathbb{R}$ containing 4 and a $\mathcal{C}^1$ function $f\colon U \longrightarrow \mathbb{R}^2$ such that $f(4) = (-1,2)$ and such that

$$F(x, f(x)) = 0,$$

for every $x \in U$.

(b) Find the derivative $Df(4)$.

5. (10 pts) (a) Show that, in a neighbourhood of $(2,-1,1)$, the subset

$$S = \{\, (x,y,z) \in \mathbb{R}^3 \mid x^3y^3 + y^3z^3 + z^3x^3 = -1 \,\},$$

is the graph of a $\mathcal{C}^1$ function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$.

(b) Determine

$$\frac{\partial f}{\partial x}(2,-1) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(2,-1).$$

6. (10 pts) (2.5.9).
7. (10 pts) (2.5.11).
8. (10 pts) (2.5.22).
9. (10 pts) (2.6.21)
10. (10 pts) (2.6.24).

Just for fun: For any $\epsilon > 0$, find an example of a smooth function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that for $f(x) = 0$, if $x \leq 0$ or $x \geq 1$ and $f(x) = 1$ if $\epsilon < x < 1 - \epsilon$. (*Hint: consider the function* $g(x) = e^{-1/x^2}$.)