TWELFTH HOMEWORK, PRACTICE PROBLEMS

1. Let $\vec{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the vector field given by

$$\vec{F}(x,y,z) = ay^2\hat{\imath} + 2y(x+z)\hat{\jmath} + (by^2+z^2)\hat{k}.$$

(i) For which values of a and b is the vector field $\vec{F}$ conservative?
(ii) Find a function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\vec{F} = \operatorname{grad} f$, for these values.
(iii) Find the equation of a surface S with the property that for every smooth oriented curve C lying on S,

$$\int_C \vec{F}\cdot \mathrm{d}\vec{s} = 0,$$

for these values.
2. Let S be the rectangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,2,2)$ and $(0,2,2)$. Find the flux of the vector field $\vec{F}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, given by

$$\vec{F}(x,y,z) = y^2\hat{\imath} - z^2\hat{\jmath} + x^2\hat{k},$$

through S in the direction of the unit normal vector $\hat{n}$, for which $\hat{n}\cdot\hat{k} > 0$.
3. Let $C_a(P)$ be the circle of radius a centered at P and oriented counter-clockwise. A smooth rotation free vector field $\vec{F}$ is defined on the whole of $\mathbb{R}^2$, except for the points $P_0 = (0,0)$, $P_1 = (4,0)$, and $P_3 = (8,0)$, and

$$\int_{C_2(P_0)} \vec{F}\cdot\mathrm{d}\vec{s} = -2, \qquad \int_{C_6(P_0)} \vec{F}\cdot\mathrm{d}\vec{s} = 1 \qquad \text{and} \qquad \int_{C_{10}(P_0)} \vec{F}\cdot\mathrm{d}\vec{s} = 3.$$

Find the following line integrals.
(a)

$$\int_{C_1(P_1)} \vec{F}\cdot\mathrm{d}\vec{s}.$$

(b)

$$\int_{C_1(P_2)} \vec{F}\cdot\mathrm{d}\vec{s}.$$

(c)

$$\int_{C_6(P_2)} \vec{F}\cdot\mathrm{d}\vec{s}.$$

4. (6.3.16)
5. (7.1.4)

6. (7.1.20)
7. (7.2.13)
9. (7.2.17)
10. (7.3.11)
11. (7.3.13)
12. (7.3.16)
13. (7.3.18)
14. (7.3.19)

Just for fun: Let ω be a k-form on $\mathbb{R}^n$. Show that

$$\mathrm{d}(\mathrm{d}\omega) = 0.$$

This basic fact about d is often expressed in the formula $\mathrm{d}^2 = 0$.