TENTH HOMEWORK, DUE THURSDAY NOVEMBER 18TH

Feel free to work with others, but the final write-up should be entirely your own and based on your own understanding.

1. (10 pts) In this problem we will check that

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

Let

$$I = \int_{0}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

(i) Show that

$$I^2 \geq \frac{\pi}{4},$$

using the fact that the quartercircle

$$x^2 + y^2 \leq a^2 \qquad x \geq 0 \qquad \text{and} \qquad y \geq 0,$$

is a subset of the square $[0, a] \times [0, a]$.

(ii) Show that

$$I^2 \leq \frac{\pi}{4},$$

using the fact that the quartercircle

$$x^2 + y^2 \leq 2a^2 \qquad x \geq 0 \qquad \text{and} \qquad y \geq 0,$$

contains the square $[0, a] \times [0, a]$.

(*Hint: in both cases, write down an inequality between the two integrals and take the limit as a goes to ∞.)*

2. (10 pts) (5.5.15)
3. (10 pts) (5.5.16)
4. (10 pts) (5.5.20)
5. (10 pts) (5.5.21)
6. (10 pts) (5.5.22)
7. (10 pts) (5.5.29)
8. (10 pts) (5.5.30)
9. (10 pts) (5.5.31)
10. (10 pts) (5.5.32)

Just for fun: Given a subset A of $\mathbb{R}^n$, we say that $a \in \mathbb{R}^n$ is a **strict limit point** if a is a limit point of $A - \{a\}$ (or what comes to the same thing, if there is a sequence $a_1, a_2, \dots$ of *distinct* elements of A with

limit a). For what we are going to be interested in, we may assume that $n = 1$.
By way of illustration, the set of strict limit points of

$$A = \{ \frac{1}{n} \,|\, n \in \mathbb{N} \} \cup \{0\},$$

is $\{0\}$. Note that this is also the set of strict limit points of

$$A - \{0\} = \{ \frac{1}{n} \,|\, n \in \mathbb{N} \}.$$

Starting with any subset $A \subset \mathbb{R}$, we are interested in how many times we need to iterate the procedure of passing to the set of strict limit points, until we get to the emptyset.
For example, starting with

$$A_0 = \{ \frac{1}{n} \,|\, n \in \mathbb{N} \} \cup \{0\},$$

if we replace A_0 by its set of strict limit points, then we get

$$A_1 = \{0\}.$$

The set of limit points of A_1 is the emptyset,

$$A_2 = \varnothing,$$

so that it takes two steps. Find examples of sets $A \subset \mathbb{R}$ such that it takes n steps to get to the empty set. Find an example of a set A which takes infinitely many steps. Now find another set A' which takes more steps than A to get to the empty set. (It is probably not so instructive to write down these examples, it is more interesting to convince yourself that they exist; for this it is helpful to sketch points the number line).