

MODEL ANSWERS TO HWK #8

1. (i) Call a line **standard** if it is either horizontal or vertical.

It is expedient to prove an even stronger result. We prove that if $f: U \rightarrow \mathbb{C}$ is any function, where U is the complement of finitely many standard lines, which restricts to a polynomial on any standard line contained in U , then f is a polynomial. We will be somewhat sloppy and say that a standard line is contained in U if it is not one of the deleted lines (strictly speaking, only the line minus finitely many points lies in U).

Note that if $V \subset U$ is obtained from U by deleting finitely many more standard lines and $f|_V$ is a polynomial, then f is a polynomial. Indeed $f|_V$ extends to a polynomial function $g: U \rightarrow \mathbb{C}$. If l is a line in U then $f|_l$ and $g|_l$ agree on an open subset of the line and so are equal. But then $f = g$.

Let d be the smallest positive integer such that there are uncountably many real numbers r such that the restriction of f to the vertical line $x = r$ is a polynomial of degree at most d and there are uncountably many real numbers s such that the restriction of f to the horizontal line $y = s$ is a polynomial of degree at most d .

We proceed by induction on d . Suppose that $d < 0$, so that $f(x, y)$ restricts to the zero function on infinitely many horizontal and infinitely many vertical lines. If l is any standard line contained in U then the restriction of f to l is a polynomial with infinitely many zeroes, so that f must be the zero function, which is represented by the zero polynomial.

Suppose that $d \geq 0$. Note that the change of coordinates $x \rightarrow x - a$ does not change the property that U is the complement of finitely many standard lines, that f restricted to any standard line is a polynomial and it also does not change the value of d . So we might as well assume that the x -axis is contained in U and $f(x, 0)$ is a polynomial of degree at most d . Let $g(x, y) = f(x, y) - f(x, 0)$. Then the restriction of $g(x, y)$ to every vertical line is a polynomial in y which vanishes at the origin. Let $V \subset U$ be the set obtained by deleting the line $y = 0$. Let

$$h: V \rightarrow \mathbb{C},$$

be the function $h(x, y) = f(x, y)/y$. Then V is obtained from \mathbb{C}^2 by deleting finitely many standard lines, $h(x, y)$ is a function which when restricted to any standard line in V is a polynomial, which has degree

at most $d-1$ on uncountably many standard lines. By induction $h(x, y)$ is a polynomial function. It follows that $f(x, y) = yh(x, y) + f(x, 0)$ is a polynomial function on V , whence on U . Thus $P(\mathbb{C})$ is true.

(ii) Enumerate, c_1, c_2, \dots the points of $\overline{\mathbb{Q}}$ and let $h_n(y)$ (respectively $v_n(x)$) be the polynomial which vanishes on the first n horizontal (respectively vertical) lines. Let

$$f(x, y) = \sum_{i=0}^{\infty} h_i(y)v_i(x).$$

It is clear that $f(x, y)$ is not a polynomial. But suppose we pick a horizontal line, given by, $y = b$. Then $b = c_n$ for some n and so

$$f(x, b) = \sum_{i \leq n} h_i(b)v_i(x),$$

so that $f(x, b)$ is a polynomial. By symmetry the restriction of $f(x, y)$ to any vertical line is a polynomial. So $P(\overline{\mathbb{Q}})$ fails.

(iii) Clear, from (i) and (ii) and the Lefschetz principle.

2. (i) Let $\phi \in V$, $\psi \in V$ and let $\lambda \in k$. Then

$$\phi: z \longrightarrow X,$$

is a morphism of schemes over k , such that the unique point of z goes to x . But then ϕ corresponds to a morphism of local rings over k ,

$$f: \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

Similarly suppose that ψ corresponds to g . Note that the function

$$m_\lambda: \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \quad \text{given by} \quad a + b\epsilon \longrightarrow a + \lambda b\epsilon,$$

is a morphism of local rings, which is an isomorphism if and only if $\lambda \neq 0$. Let $\lambda\phi$ be the morphism of schemes corresponding to the morphism of local rings $m_\lambda \circ f$. Similarly, define a map

$$\alpha: \frac{k[\epsilon_1]}{\langle \epsilon_1^2 \rangle} \otimes_k \frac{k[\epsilon_2]}{\langle \epsilon_2^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

by sending both ϵ_1 and ϵ_2 to ϵ and extend by linearity to get a morphism of local rings. Composing with the natural map

$$(f, g): \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon_1]}{\langle \epsilon_1^2 \rangle} \otimes_k \frac{k[\epsilon_2]}{\langle \epsilon_2^2 \rangle},$$

we get a morphism of local rings and this defines a morphism

$$\phi + \psi: z \longrightarrow X.$$

This defines an operation of scalar multiplication and addition of vectors, which clearly satisfy the axioms for a vector space.

(ii) If $\phi \in T_x X$ and

$$f: \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

is the corresponding morphism of local rings, then the kernel of f contains \mathfrak{m}^2 . On the other hand, the inverse image of $\langle \epsilon \rangle$ is by definition contained in \mathfrak{m} . It follows that we get a linear map of vector spaces

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow k\langle \epsilon \rangle \simeq k,$$

that is, an element of the dual space

$$\left(\frac{\mathfrak{m}}{\mathfrak{m}^2} \right)^*,$$

and it is not hard to see that this assignment induces a bijection.

2. 1. By assumption there are open subsets U and V and isomorphisms $\mathcal{L}|_U \simeq \mathcal{O}_U$, $\mathcal{M}|_V \simeq \mathcal{O}_V$. Passing to the open subset $U \cap V$ we may as well assume that $\mathcal{L} = \mathcal{M} = \mathcal{O}_X$. It suffices to check that the map is an isomorphism on stalks. Suppose that $x \in X$ and let $A = \mathcal{O}_{X,x}$. Then A is a local ring and we are given a surjective A -module homomorphism $\phi: A \longrightarrow A$. ϕ is given by multiplication by an element a of A . Suppose that $\phi(b) = 1$. Then $ab = 1$ and so a is a unit and ϕ is an isomorphism. Thus f is an isomorphism on stalks and f is an isomorphism.

2. Suppose that $m > n$. As $\dim V \leq n + 1$ it follows that t_i is a linear combination of the other sections, for some $1 \leq i \leq m$. Let $\pi: \mathbb{P}^m \longrightarrow \mathbb{P}^{m-1}$ be the projection map which drop the i th coordinate. The composition

$$\pi \circ \phi: X \longrightarrow \mathbb{P}^{m-1},$$

is the morphism given by $t_0, t_1, \dots, \hat{t}_i, \dots, t_n$. So we may assume $m = n$ by induction on $m - n$.

Suppose first that $\dim |V| = \dim V - 1 = n$. In this case both s_1, s_2, \dots, s_n and t_1, t_2, \dots, t_n are bases of V . So there is a unique matrix $A = (a_{ij})$ such that

$$t_i = \sum a_{ij} s_j.$$

This matrix corresponds to an isomorphism $\sigma: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ and it is clear that $\psi = \sigma \circ \phi$.

In general the image of X is contained in linear spaces Λ_i , $i = 1$ and 2 of dimension $\dim |V| = \dim V - 1$. Pick complimentary linear subspaces Λ'_i . We have already exhibited an isomorphism $\sigma_1: \Lambda_1 \longrightarrow \Lambda_2$, such that $\psi = \sigma_1 \circ \phi$ and we may extend this to an isomorphism of $\sigma: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\sigma(\Lambda'_1) = \Lambda'_2$ and $\psi = \sigma \circ \phi$.

3. (a) Let $\mathcal{L} = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$. As $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$ it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, for some integer d . As \mathcal{L} is globally generated $d \geq 0$. If $d = 0$ then $\phi(\mathbb{P}^n)$ is a point. Otherwise $d > 0$ and \mathcal{L} is ample. Suppose that $C \subset \mathbb{P}^n$ is an irreducible curve. As \mathcal{L} is ample, $\mathcal{L}|_C$ is not the trivial invertible sheaf. If $x \in C$ then we may find a section $\sigma \in H^0(\mathbb{P}^n, \mathcal{L})$ which does not vanish at x . As $\mathcal{L}|_C$ is not the trivial invertible sheaf, $\sigma|_C$ must vanish somewhere. Therefore the image of C is curve. Let $X = \phi(\mathbb{P}^n)$. If $\dim X < n$, then the fibres of $\phi: \mathbb{P}^n \rightarrow X$ are positive dimensional. But then the fibres must contain curves C (just cut by hyperplanes) which are sent to a point, a contradiction.
- (b) As stated, this is obviously false. Let $\phi: \mathbb{P}^1 \rightarrow \mathbb{P}^2$ be the morphism

$$[S : T] \rightarrow [S : S : T].$$

It is clear in this case that $d = 1$. The 1-uple embedding is the identity. But then we cannot hope to project from \mathbb{P}^1 down to \mathbb{P}^2 .

So let's assume that the image of ϕ is non-degenerate, that is, not contained in a hyperplane. ϕ is given by a linear system. It follows that there is an invertible sheaf \mathcal{L} and a collection of sections $s_1, s_2, \dots, s_a \in H^0(\mathbb{P}^n, \mathcal{L})$. Since $\text{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^n}(1)$, it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, up to isomorphism. Let t_0, t_1, \dots, t_N be the standard basis of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ given by monomials of degree d . Then the induced morphism is the d -uple embedding $\mathbb{P}^n \rightarrow \mathbb{P}^N$. Let

$$V \subset H^0(\mathbb{P}^n, \mathcal{L}),$$

be the subvector space spanned by s_1, s_2, \dots, s_a . Our assumption that ϕ is non-degenerate means that s_1, s_2, \dots, s_a are a basis of V . We may extend this to a basis of $H^0(\mathbb{P}^n, \mathcal{L})$ and this defines an automorphism σ of \mathbb{P}^N . Projecting down to the first $a+1$ coordinates gives the morphism ϕ . Finally note that applying an automorphism of \mathbb{P}^N is the same as projecting from the linear space L , which is the image under σ of the space spanned by the last $N - a - 1$ coordinates and an automorphism of \mathbb{P}^n .

4. (a) If \mathcal{L} is ample then \mathcal{L}^m is very ample, for some positive integer m . But then there is an immersion $X \rightarrow \mathbb{P}_k^n$ for some positive integer n and it follows that X is separated.

(b) By assumption there are two open subsets U_1 and U_2 both of which are isomorphic to \mathbb{A}_k^1 . Let \mathcal{L} be an invertible sheaf on X and let \mathcal{L}_i be the restriction of \mathcal{L} to U_i . As $\text{Pic}(U_i) = 0$ it follows that $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$. Suppose that $\{p_1, p_2\}$ are the double points of X so that

$$X - \{p_1, p_2\} = U_i - \{p_i\}.$$

The section 1 on U_1 corresponds to a non-vanishing section $f(x)$ on U_2 . It follows that $f(x) = ax^m$, for some positive integer m and a non-zero scalar a . Multiplying through by automorphisms of U_2 which fix p_2 we can assume that $a = 1$. Let's call this invertible sheaf \mathcal{L}_m . If we tensor \mathcal{L}_m with \mathcal{L}_n we get the global section 1 on U_1 and the global section $f(x) = x^{m+n}$ on U_2 . It follows that $\text{Pic}(X) = \mathbb{Z}$ (the inverse of the \mathcal{L}_m is the invertible sheaf \mathcal{L}_{-m} which has a global section which restricts to x^m on U_1 and 1 on U_2).

Now let's consider if any of these line bundles are ample. By symmetry we may suppose that $m \geq 0$. Sections of \mathcal{L}_m correspond to pairs $g(x)$ on U_1 and $x^m g(x)$ on U_2 , where $g(x)$ is a polynomial. There are two cases. If $m > 0$ then this section always vanishes at p_2 . If $m = 0$ then this section only vanishes at p_1 if $g(x)$ has a zero at p_1 , in which case the section also vanishes at p_2 . Either way, \mathcal{L}_m does not separate points.

5. (a) Let \mathcal{F} be a coherent sheaf. By assumption there is an integer n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for all $n \geq n_0$. Pick $x \in X$. Then we may find $l_1, l_2, \dots, l_k \in H^0(X, \mathcal{F} \otimes \mathcal{L}^{n_0})$ whose images generate the stalk at x . Pick $m \in \mathcal{M}$ not vanishing at x . Then $m^{n_0} l_1, m^{n_0} l_2, \dots, m^{n_0} l_k$ are naturally global sections of $\mathcal{F} \otimes \mathcal{L}^{n_0} \otimes \mathcal{M}^{n_0}$ which generate the stalk at x . Hence $\mathcal{F} \otimes \mathcal{L}^{n_0} \otimes \mathcal{M}^{n_0}$ is globally generated so that $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) As \mathcal{L} is ample, we may pick l so that $\mathcal{M} \otimes \mathcal{L}^l$ is globally generated. If $m > 0$ is any positive integer, then

$$\mathcal{M} \otimes \mathcal{L}^{l+m} = \mathcal{M} \otimes \mathcal{L}^l \otimes \mathcal{L}^m,$$

is ample by (a). So $\mathcal{M} \otimes \mathcal{L}^n$ is ample for any $n > l$.

(c) Since \mathcal{O}_X is globally generated we may find $k > 0$ so that \mathcal{M}^k is globally generated. As \mathcal{L} is ample then so is \mathcal{L}^k . But then

$$(\mathcal{L} \otimes \mathcal{M})^k = \mathcal{L}^k \otimes \mathcal{M}^k,$$

is ample by (a). It follows that

$$\mathcal{L} \otimes \mathcal{M},$$

is ample.

(d) By assumption we may find sections $l_1, l_2, \dots, l_a \in H^0(X, \mathcal{L})$ and $m_1, m_2, \dots, m_b \in H^0(X, \mathcal{M})$ such that X_{l_i} and X_{m_j} are an open affine cover of X . Consider the sections $l_i m_j \in H^0(X, \mathcal{L} \otimes \mathcal{M})$. Note that $X_{ij} = X_{l_i} \cap X_{m_j}$ is affine. Since m_j is not zero on X_{ij} , the images

$$\frac{l_{i'} m_j}{l_i m_j} = \frac{l_{i'}}{l_i},$$

generate $H^0(X_{ij}, \mathcal{O}_X)$, since the images even generate $H^0(X_{l_i}, \mathcal{O}_X)$. It follows that the sections $l_i m_j$ define an immersion of X into \mathbb{P}^n into projective space such that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ is $\mathcal{L} \otimes \mathcal{M}$. But then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

(e) First of all we know that there is a positive integer m such that \mathcal{L}^m is very ample. On the other hand, by the definition of ample, we know that there is an integer m_0 such that \mathcal{L}^n is globally generated for all $n \geq m_0$. Let $n_0 = m_0 + m$. If $n \geq n_0 + m$ then

$$\mathcal{L}^n = \mathcal{L}^{n-m} \otimes \mathcal{L}^m,$$

is very ample by (d).