MODEL ANSWERS TO HWK #6

1. A dense open subset is given by taking l distinct reduced points. For each point we have ∞^n choices (any point of X) and so the dimension is nl.
2. (i) Let $R = k[x_1, x_2, \dots, x_n]$. Then

$$\dim_k \frac{R}{\mathfrak{m}^{p+1}},$$

is the number of monomials of degree at most p in n variables, which is the same as the number of monomials of degree p in $n+1$ variables. The usual stars and bars argument says that this is

$$\binom{n+p}{n}.$$

So the length of z is

$$l = \binom{n+p}{n} - q.$$

(ii) Clear, since we just define the product of any element of W with any homogeneous element of degree at least one to be zero.
(iii) The dimension of V is equal to

$$m = \binom{n+p-1}{n-1},$$

the number of monomials of degree p in n variables. The dimension of the Grassmannian of q planes in the vector space of planes in V is then $q(m-q)$. Thus the space of such zero dimensional schemes has dimension

$$q(m-q).$$

3. Varying the support, we get dimension

$$n + q(m-q).$$

If we fix p and n then we want to maximise

$$n + q(m-q).$$

Clearly we should take $q = \llcorner m/2 \lrcorner$. In order to simplify the notation, let's assume that m is even, so that $q = m/2$.

Let us take $n = 3$. Then the dimension of the space of curvilinear schemes is

$$3l = 3\binom{3+p}{3} - 3/2\binom{p+2}{2},$$

and the dimension of the space of schemes from (iii) is

$$3 + 1/4\binom{p+2}{2}^2.$$

Therefore, if p is large enough then we win, since the first expression is a polynomial of degree 3 in p and the second expression is a polynomial of degree 4 in p. It is still interesting to figure out exactly when this happens; in fact $p = 7$ will do, in which case the length is 102. In $\mathbb{A}^4_k$, $p = 3$ will do.

4. (i) A closed immersion of

$$\operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

is by definition the same as a morphism

$$\phi\colon \operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \longrightarrow X,$$

modulo isomorphisms

$$\psi\colon \operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \longrightarrow \operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

over $\operatorname{Spec} k$ and the latter are the same as local ring isomorphisms

$$f\colon \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

(Since one is sent to one, any such automatically fixes k). The group of local ring isomorphisms is given by $\{\mu_\lambda\}_{\lambda \in k^*}$, and the action on T_xX is the standard one which gives $\mathbb{P}(T_xX)$.

(ii) Let U_i be the open set of length two schemes of the form

$$\langle x_1 + a_1x_i, x_2 + a_2x_i, \dots, x_1 + a_nx_i \rangle + \mathfrak{m}^2,$$

where we omit the ith term. We first show that $\pi^{-1}(U_i)$ is isomorphic to $\mathbb{A}^{n-1}_{U_i} = \mathbb{A}^{2(n-1)}_k$ over U_i. By symmetry we may suppose that $i = n$. Let z be a length three scheme with ideal I such that $I + \mathfrak{m}^2 \in U_n$ (here we cheat a little and identify a length two scheme with its ideal). Suppose that

$$I + \mathfrak{m}^2 = \langle x_1 + a_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n \rangle + \mathfrak{m}^2.$$

I claim that any such is uniquely of the form,

$$I = \langle x_1+a_1x_n+b_1x_n^2, x_2+a_2x_n+b_2x_n^2, \dots, x_{n-1}+a_{n-1}x_n+b_{n-1}x_n^2\rangle+\mathfrak{m}^3.$$

If we project this scheme down to any of the coordinate hyperplanes, this is the same as dropping the corresponding variable. Uniqueness is then clear by induction on n. On the other hand any ideal I such that

$$I + \mathfrak{m}^2 = \langle x_1 + a_1x_n, x_2 + a_2x_n, \dots, x_{n-1} + a_{n-1}x_n\rangle + \mathfrak{m}^2.$$

contains quadratic monomials of the form $x_ix_j + a_ix_jx_n$, where $1 \leq i \leq n-1$ and $1 \leq j \leq n$. It is then clear that we can put find $b_1, b_2, \dots, b_{n-1}$ such that

$$I = \langle x_1+a_1x_n+b_1x_n^2, x_2+a_2x_n+b_2x_n^2, \dots, x_{n-1}+a_{n-1}x_n+b_{n-1}x_n^2\rangle+\mathfrak{m}^3.$$

There is then an obvious isomorphism $\pi^{-1}(U_i) \longrightarrow \mathbb{A}^{n-1}_{U_i}$.
Now consider what happens on overlaps. Suppose that $I \in U_1 \cap U_n$. Then

$$I = \langle x_1+a_1x_n+b_1x_n^2, x_2+a_2x_n+b_2x_n^2, \dots, x_{n-1}+a_{n-1}x_n+b_{n-1}x_n^2\rangle+\mathfrak{m}^3.$$

and also

$$I = \langle x_2 + a'_2x_1 + b'_2x_1^2, x_3 + a'_3x_1 + b'_3x_1^2, \dots, x_n + a'_nx_1 + b'_nx_1^2\rangle + \mathfrak{m}^3.$$

By assumption $a_1 \neq 0$. Therefore

$$1/a_1(x_1 + a_1x_n + b_1x_n^2) = x_n + (1/a_1)x_1 + (b_1/a_1)x_n^2.$$

It follows that $a'_n = 1/a_1$. On the other hand,

$$(x_i+a_ix_n+b_ix_n^2)-a_i(x_1+a_1x_n+b_1x_n^2)/a_1 = x_i-(a_i/a_1)x_1+(b_ia_1-b_1a_i)/a_1x_n^2.$$

Therefore $a'_i = -a_i/a_1$.

$$(x_1^2 + a_1x_1x_n) - a_1(x_1x_n + a_1x_n^2) = x_1^2 - a_1^2x_n^2.$$

Therefore $x_n^2 = x_1^2/a_1^2$ modulo I. It follows that

$$x_n + (1/a_1)x_1 + (b_1/a_1)x_n^2 = x_n + (1/a_1)x_1 + (b_1/a_1^3)x_1^2,$$

and

$$x_i-(a_i/a_1)x_1+(b_ia_1-b_1a_i)/a_1x_n^2 = x_i-(a_i/a_1)x_1+(b_ia_1-b_1a_i)/a_1^3x_n^2,$$

modulo I. In particular the transition functions are linear in $b_1, b_2, \dots, b_{n-1}$ and so π is an affine bundle.
(iii) A length two scheme is determined by the line which contains it. Given a line we get a unique length n curvilinear scheme. This gives a section of π. Algebraically, given a curvilinear scheme of length two, determined by the ideal $\langle x_1, x_2, \dots, x_{n-1}\rangle + \mathfrak{m}^2$, let σ be the curvilinear scheme of length three given by $\langle x_1, x_2, \dots, x_{n-1}\rangle + \mathfrak{m}^3$.

Since the transition functions are linear, they fix this section and so π is a vector bundle.