MODEL ANSWERS TO HWK #4

5.1 (a) First we define a morphism

$$\mathcal{E} \longrightarrow \check{\check{\mathcal{E}}}.$$

Suppose we are given an open subset $U \subset X$ and a section $s \in \mathcal{E}(U)$. We want to define a sheaf homomorphism

$$\check{s} \colon \check{\mathcal{E}}|_U \longrightarrow \mathcal{O}_U.$$

Pick $V \subset U$ open. An element of $\phi \in \check{\mathcal{E}}(V)$ is a sheaf homomorphism

$$\phi \colon \mathcal{E}|_V \longrightarrow \mathcal{O}_V.$$

Then

$$\check{s}(\phi) = \phi(s|_V).$$

This defines a morphism of sheaves. To check that this morphism is an isomorphism, it suffices to check that is an isomorphism locally. So we may assume that $\mathcal{E}$ is free. As taking curly Hom commutes with direct sum, we may assume that $\mathcal{E} = \mathcal{O}_X$, in which case the result is clear.

(b) Suppose that we are given an open subset $U \subset X$. Define an $\mathcal{O}_X(U)$-module homomorphism

$$\check{\mathcal{E}}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{F}(U) \longrightarrow \mathbf{Hom}(\mathcal{E}, \mathcal{F})(U),$$

by sending $\phi \otimes \tau$ to the sheaf homomorphism

$$\psi \colon \mathcal{E}|_U \longrightarrow \mathcal{F}|_U,$$

which sends $s \in \mathcal{E}(V)$, to $\phi|_V(s)(\tau|_V)$, where $V \subset U$ is open, and then extending linearly. This defines a morphism of presheaves; by the universal property of the sheaf associated to the presheaf, this defines a morphism of sheaves,

$$\check{\mathcal{E}} \underset{\mathcal{O}_X}{\otimes} \mathcal{F} \longrightarrow \mathbf{Hom}(\mathcal{E}, \mathcal{F}).$$

To check that this morphism is an isomorphism, it suffices to check this locally. To check this, we may assume that $\mathcal{E}$ is free. Since taking curly Hom commutes with direct sum, we may assume that $\mathcal{E}$ has rank one, so that $\mathcal{E} \simeq \mathcal{O}_X$. In this case, both sides are isomorphic to $\mathcal{F}$ and the given morphism is an isomorphism.

(c) Suppose we are given a morphism of sheaves

$$\phi \colon \mathcal{F} \longrightarrow \check{\mathcal{E}} \otimes \mathcal{G}.$$

Let $\mathcal{L}$ be the presheaf
$$U \longrightarrow \check{\mathcal{E}}(U) \otimes \mathcal{G}(U).$$
By the universal property of the sheaf associated to the presheaf, we get a morphism of presheaves
$$\psi\colon \mathcal{G} \longrightarrow \mathcal{L}.$$
We want to define a morphism of sheaves
$$\sigma\colon \mathcal{E} \otimes \mathcal{F} \longrightarrow \mathcal{G}.$$
Let $\mathcal{K}$ be the presheaf
$$U \longrightarrow \mathcal{E}(U) \otimes \mathcal{F}(U).$$
To define σ is the same as to define
$$\tau\colon \mathcal{K} \longrightarrow \mathcal{G}.$$
Let $U \subset X$ be an open subset and let $e \otimes f \in \mathcal{K}(U)$. Suppose that $\psi(f) = \sum_i e_i \otimes g_i$. Define
$$\tau(e \otimes f) = \sum e_i(e) g_i,$$
and extend this linearly.
Similarly we may define a morphism the other way, which is the inverse of the assignment $\phi \longrightarrow \tau$.
(d) Let $\mathcal{G} = \check{\mathcal{E}}$. Note that
$$\begin{aligned}
\Gamma(X, f_*(\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{E}) &= \mathrm{Hom}(\mathcal{O}_X, f_*(\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{E})) \\
&= \mathrm{Hom}(\mathcal{O}_Y, \mathcal{F} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{E}) \\
&= \mathrm{Hom}(\mathcal{O}_Y, \mathbf{Hom}(f^*\mathcal{G}, \mathcal{F})) \\
&= \mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \\
&= \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \\
&= \mathrm{Hom}(\mathcal{G} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X, f_*\mathcal{F}) \\
&= \mathrm{Hom}(\mathcal{O}_X, f_*\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}) \\
&= \Gamma(X, f_*\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}).
\end{aligned}$$
Let $U \subset X$ be an open subset. As we have constructed a natural isomorphism
$$\Gamma(U, f_*(\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{E}) \simeq \Gamma(U, f_*\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}),$$

we have an isomorphism of sheaves

$$f_*(\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{E}) \simeq f_*\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{E}.$$

5.4 One direction is clear; if $\mathcal{F}$ is locally the cokernel of a morphism between two locally free sheaves, then $\mathcal{F}$ is quasi-coherent, since the property of being quasi-coherent is local, locally free sheaves are quasi-coherent, and the cokernel of a morphism between two quasi-coherent sheaves is quasi-coherent. If in addition X is Noetherian and the locally free sheaves have finite rank, then $\mathcal{F}$ is coherent.
Now suppose that $\mathcal{F}$ is quasi-coherent. Passing to an open affine subset we may assume that $X = \operatorname{Spec} A$ and $\mathcal{F} = \tilde{M}$ is the sheaf associated to an A-module M. If we pick a set of generators for M as an A-module, then we may find a free A-module F and a surjective A-module homomorphism $F \longrightarrow M$. Let R be the kernel. Then R is also a free A-module. It follows that $\mathcal{F}$ is isomorphic to the cokernel of the morphism of locally free sheaves $\tilde{R} \longrightarrow \tilde{F}$. If $\mathcal{F}$ is coherent then we may choose F of finite rank, in which case R has finite rank as well.
5.5. (a) Let $X = \mathbb{A}^1_K$ and $Y = \mathbb{A}^1_K - \{0\}$ and let $f\colon Y \longrightarrow X$ be the natural open immersion. $\mathcal{F} = \mathcal{O}_Y$ is surely coherent. $f_*\mathcal{F}$ is quasi-coherent, the sheaf associated to the $K[x]$-module $K[x, x^{-1}]$, which is clearly not a finitely generated $K[x]$-module. In fact $K[x, x^{-1}]_{\langle x \rangle}$ is not a finitely generated $K[x]_{\langle x \rangle}$-module. But then there is no open subset $U \subset \mathbb{A}^1_k = X$ containing the origin, about which $f_*\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.
(b) Let $f\colon X \longrightarrow Y$ be a closed immersion. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

where $\mathcal{I}_Y$ is quasi-coherent. Let $U = \operatorname{Spec} A \subset X$ an open affine subset. Let $I = \Gamma(U, \mathcal{O}_X) \subset A$. Then $Y \cap U = f^{-1}(U)$ is the affine scheme associated to A/I. As A/I is a finitely generated A-module, it follows that f is a finite morphism.
(c) Since the result is local on the base we may assume that $Y = \operatorname{Spec} B$ is affine. But then, since f is finite, $X = \operatorname{Spec} A$ is also affine and A is a finitely generated B-module. As $\mathcal{F}$ is coherent and X is Noetherian, there is a finitely generated A-module M such that $\mathcal{F} = \tilde{M}$.
By assumption there are $m_1, m_2, \dots, m_p \in M$ and $a_1, a_2, \dots, a_q \in A$ such that if $m \in M$ and $a \in A$ we may write

$$m = c_1 m_1 + \dots + c_p m_p \qquad \text{and} \qquad a = b_1 a_1 + dots + b_q a_q,$$

where $c_1, c_2, \dots, c_p \in A$ and $b_1, b_2, \dots, b_q \in B$. For every $c_j \in A$ we may find $b_{ij} \in B$ such that $c_j = \sum_i b_{ij} a_i$. But then

$$m = \sum_{ij} b_{ij}(a_i m_j).$$

It follows that $\{a_i m_j\}$ generate M as a B-module so that M is a finitely generated B-module. But as $f_*\mathcal{F} = \tilde{M}$, it follows that $f_*\mathcal{F}$ is coherent.
5.7. (a) By assumption there is an open affine neighbourhood $U = \operatorname{Spec} A$ of $x \in X$ and a finitely generated A-module M such that $\mathcal{F} = \tilde{M}$. As $\mathcal{F}_x$ is a finitely generated free $\mathcal{O}_{X,x}$-module, there is an isomorphism

$$\alpha\colon \bigoplus_{i=1}^{n} \mathcal{O}_{X,x} \longrightarrow \mathcal{F}_x.$$

Let $f_1, f_2, \dots, f_n \in \mathcal{F}_x$ be the images of the standard generators of $\bigoplus_{i=1}^{n} \mathcal{O}_{X,x}$. For each i we may find $x \in U_i \subset U$ open and $\sigma_i \in \mathcal{F}(U_i)$ such that f_i is represented by (σ_i, U_i). Let V be the intersection of $U_1, U_2, \dots, U_n$. Replacing σ_i by $\sigma_i|_V$ we may suppose that f_i is represented by (σ_i, V). We get a morphism of sheaves

$$\phi\colon \bigoplus_{i=1}^{n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V,$$

by sending $(g_1, g_2, \dots, g_n) \in \bigoplus_{i=1}^{n} \mathcal{O}_V(W)$ to $\sum g_i \sigma_i|_W$, where $W \subset V$ is an open subset. Clearly $\phi_x = \alpha$. Let $\mathcal{K}$ and $\mathcal{H}$ be the kernel and cokernel of ϕ. As ϕ is a morphism of coherent sheaves and X is Noetherian, it follows that $\mathcal{K}$ and $\mathcal{H}$ are coherent sheaves. As $\mathcal{K}_x = \mathcal{H}_x = 0$ it follows that there is an open neighbourhood W of x in V such that $\mathcal{K}|_U = \mathcal{H}|_U = 0$. But then $\phi|_W$ is an isomorphism of sheaves and so $\mathcal{F}|_W$ is locally free.
(b) If $\mathcal{F}$ is locally free then its stalks are obviously locally free. The converse is an easy consequence of (a).
(c) Suppose that $\mathcal{F}$ is locally free of rank one. Let $\mathcal{G} = \check{\mathcal{F}}$. Then $\check{\mathcal{G}} = \mathcal{F}$ and so

$$\mathcal{F} \otimes \mathcal{G} \simeq \mathbf{Hom}(\mathcal{F}, \mathcal{F}).$$

Define a morphism

$$\mathcal{O}_X \longrightarrow \mathbf{Hom}(\mathcal{F}, \mathcal{F}),$$

by sending f to the sheaf homorphism $\sigma \longrightarrow f|_V \sigma$, where $\sigma \in \mathcal{F}(V)$ and V is an open subset. To check that this morphism is an isomorphism we may work locally, in which case we may assume that $\mathcal{F} \simeq \mathcal{O}_X$ and the result is clear.

Now suppose that there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{O}_X$. To see that $\mathcal{F}$ is invertible of rank one, it suffices to check that

$$\mathcal{F}_x \simeq \mathcal{O}_{X,x}$$

Let $A = \mathcal{O}_{X,x}$, $M = \mathcal{F}_x$ and $N = \mathcal{G}_x$. Then M and N are A-modules, and

$$M \underset{A}{\otimes} N \simeq A.$$

Let $\mathfrak{m}$ be the unique maximal ideal of A. Then $V = M/\mathfrak{m}M$ and $W = N/\mathfrak{m}N$ are vector spaces over the field $K = A/\mathfrak{m}$. Clearly V has dimension one, as

$$V \underset{K}{\otimes} W \simeq K.$$

But then Nakayama's Lemma implies that M has rank one and so $\mathcal{F}$ is an invertible sheaf.
5.16. (a) It suffices to prove this locally, in which case we may assume that

$$\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_X.$$

The result follows easily from the corresponding result for modules of the form

$$\bigoplus_{i=1}^{r} A.$$

(b) There is a natural morphism of sheaves

$$\wedge^r \mathcal{F} \otimes \wedge^{n-r} \mathcal{F} \longrightarrow \wedge^n \mathcal{F}.$$

To check that this morphism is an isomorphism, it suffices to check this on stalks. So we have to check that

$$\wedge^r M \otimes \wedge^{n-r} M \longrightarrow \wedge^n M,$$

is an isomorphism, where M is a free module of rank n over the local ring A. Let $\mathfrak{m}$ be the maximal ideal. Then $V = M/\mathfrak{m}M$ is a vector space over the field $K = A/\mathfrak{m}$ of dimension n and by Nakayama's Lemma, it suffices to check that the induced linear map

$$\wedge^r V \otimes \wedge^{n-r} V \longrightarrow \wedge^n V,$$

is an isomorphism. But this is easy.
(c) We prove this by induction on r. First note that there is a surjective morphism of sheaves

$$\operatorname{Sym}^{r-1} \mathcal{F} \otimes \mathcal{F} \longrightarrow \operatorname{Sym}^r \mathcal{F}.$$

Composing with the natural morphism

$$\operatorname{Sym}^{r-1} \mathcal{F} \otimes \mathcal{F}' \longrightarrow \operatorname{Sym}^{r-1} \mathcal{F} \otimes \mathcal{F},$$

we get a morphism

$$\operatorname{Sym}^{r-1}\mathcal{F}\otimes\mathcal{F}'\longrightarrow\operatorname{Sym}^{r}\mathcal{F}.$$

By induction there is a filtration

$$\operatorname{Sym}^{r-1}\mathcal{F}=G^0\supset G^1\supset\cdots\supset G^{r-1}\supset G^r=0,$$

with successive quotients

$$G^i/G^{i+1}\simeq\operatorname{Sym}^{i}\mathcal{F}'\otimes\operatorname{Sym}^{r-1-i}\mathcal{F}''.$$

This induces a filtration

$$\operatorname{Sym}^{r-1}\mathcal{F}\otimes\mathcal{F}'=G^0\otimes\mathcal{F}'\supset G^1\otimes\mathcal{F}'\supset\cdots\supset G^{r-1}\otimes\mathcal{F}'\supset G^r=0,$$

with successive quotients isomorphic to

$$\operatorname{Sym}^{i}\mathcal{F}'\otimes\mathcal{F}'\otimes\operatorname{Sym}^{r-1-i}\mathcal{F}''.$$

Let F^{i+1} be the image of $G^i\otimes\mathcal{F}'$ inside $\operatorname{Sym}^r\mathcal{F}$. Then the successive quotients are

$$F^i/F^{i-1}\simeq\operatorname{Sym}^{i}\mathcal{F}'\otimes\operatorname{Sym}^{r-i}\mathcal{F}'',$$

as required.

(d) We prove this by induction on r. First note that there is a surjective morphism of sheaves

$$\wedge^{r-1}\mathcal{F}\otimes\mathcal{F}\longrightarrow\wedge^{r}\mathcal{F}.$$

Composing with the natural morphism

$$\wedge^{r-1}\mathcal{F}\otimes\mathcal{F}'\longrightarrow\wedge^{r-1}\mathcal{F}\otimes\mathcal{F},$$

we get a morphism

$$\wedge^{r-1}\mathcal{F}\otimes\mathcal{F}'\longrightarrow\wedge^{r}\mathcal{F}.$$

By induction there is a filtration

$$\wedge^{r-1}\mathcal{F}=G^0\supset G^1\supset\cdots\supset G^{r-1}\supset G^r=0,$$

with successive quotients

$$G^i/G^{i+1}\simeq\wedge^{i}\mathcal{F}'\otimes\wedge^{r-1-i}\mathcal{F}''.$$

This induces a filtration

$$\wedge^{r-1}\mathcal{F}\otimes\mathcal{F}'=G^0\otimes\mathcal{F}'\supset G^1\otimes\mathcal{F}'\supset\cdots\supset G^{r-1}\otimes\mathcal{F}'\supset G^r=0,$$

with successive quotients isomorphic to

$$\wedge^{i}\mathcal{F}'\otimes\mathcal{F}'\otimes\wedge^{r-1-i}\mathcal{F}''.$$

Let F^{i+1} be the image of $G^i\otimes\mathcal{F}'$ inside $\wedge^r\mathcal{F}$. Then the successive quotients are

$$F^i/F^{i-1}\simeq\wedge^{i}\mathcal{F}'\otimes\wedge^{r-i}\mathcal{F}'',$$

as required.

(e) We first prove that the tensor operations satisfy an appropriate universal property. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of of $\mathcal{O}_X$-modules. The tensor product $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}$ is a sheaf of $\mathcal{O}_X$-modules. Moreover there is a bilinear map

$$\mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G},$$

which is universal amongst all other such bilinear maps: if we are given another bilinear map

$$\mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{H},$$

where $\mathcal{H}$ is a sheaf of $\mathcal{O}_X$-modules, then there is a unique morphism of $\mathcal{O}_X$-modules

$$\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G} \longrightarrow \mathcal{H},$$

which makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{F} \oplus \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H}. \\
\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G} \arrow[ur] &
\end{tikzcd}$$

In fact the presheaf

$$U \longrightarrow \mathcal{F}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{G}(U),$$

clearly satisfies the appropriate universal property in the category of presheaves and the universal property of the sheaf associated to the presheaf implies that the tensor product satisfies the same universal property in the category of sheaves.

Now suppose we are given $f\colon X \longrightarrow Y$ a morphism of ringed spaces. Note that there is a bilinear map

$$f^*\mathcal{F} \oplus f^*\mathcal{G} \longrightarrow f^*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}),$$

using the fact that f^* and f_* are adjoint. Let $\mathcal{H}$ be sheaf of $\mathcal{O}_Y$-modules and let

$$f^*\mathcal{F} \oplus f^*\mathcal{G} \longrightarrow \mathcal{H},$$

be a bilinear map. By adjointness, we get a bilinear map

$$\mathcal{F} \oplus \mathcal{G} \longrightarrow f_*\mathcal{H}.$$

By the universal property of the tensor product there is a bilinear map

$$\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G} \longrightarrow f_*\mathcal{H}.$$

Thus we get a morphism

$$f^*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{G}) \longrightarrow \mathcal{H}.$$

It is clear that the relevant diagram commutes.