MODEL ANSWERS TO HWK #2

4.8. (d) Let $f_i \colon X_i \longrightarrow Y_i$ be two morphisms, $i = 1$, 2, having property $\mathcal{P}$. Let $f_1 \times f_2 \colon X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ be the induced morphism. If we consider the fibre square,

$$\begin{tikzcd} Y_1 \times X_2 \arrow[r] \arrow[d, "i\times f_2"'] & X_2 \arrow[d, "f_2"] \\ Y_1 \times Y_2 \arrow[r] & Y_2, \end{tikzcd}$$

where the bottom arrow is the natural projection, then we see that $Y_1 \times X_2 \longrightarrow Y_1 \times Y_2$ satisfies property $\mathcal{P}$ by (c). Now consider the fibre square

$$\begin{tikzcd} X_1 \times X_2 \arrow[r] \arrow[d, "f_1\times i"'] & X_1 \arrow[d, "f_1"] \\ Y_1 \times X_2 \arrow[r] & Y_1, \end{tikzcd}$$

where the bottom arrow is the natural projection. Then $X_1 \times X_2 \longrightarrow Y_1 \times X_2$ satisfies property $\mathcal{P}$ by (c). On the other hand, $f_1 \times f_2$ is the composition of these two morphisms and so $f_1 \times f_2$ satisfies property $\mathcal{P}$ by (b).

(e) Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta_Y"] \\ X \underset{Z}{\times} Y \arrow[r] & Y \underset{Z}{\times} Y, \end{tikzcd}$$

where the bottom row is induced from the top row and the fact that the fibre product defines a functor. Suppose that W maps to both $X \underset{Z}{\times} Y$ and Y over $Y \underset{Z}{\times} Y$. As there is a natural morphism $X \underset{Z}{\times} Y \longrightarrow X$, W maps to X by composition, and so this commutative diagram is a fibre square. As Δ_Y is a closed immersion, it satisfies property $\mathcal{P}$ by (a) and it follows that Γ_f satisfies property $\mathcal{P}$ by (c).

On the other hand, if we consider the fibre square

$$\begin{array}{ccc} X \underset{Z}{\times} Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle g\circ f} \\ Y & \longrightarrow & Z, \end{array}$$

then (c) implies that the natural morphism $X \underset{Z}{\times} Y \longrightarrow Y$ satisfies property $\mathcal{P}$.

Finally observe that f is the composition of these two morphisms, so that f satisfies property $\mathcal{P}$ by (b).

(f) Note that the following commutative diagram

$$\begin{array}{ccc} X_{\mathrm{red}} & \longrightarrow & X \\ {\scriptstyle f_{\mathrm{red}}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y_{\mathrm{red}} & \longrightarrow & Y, \end{array}$$

is a fibre square. So f_{red} satisfies property $\mathcal{P}$ by (c).

4.9. We first show that there is a closed immersion

$$\sigma \colon \mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z \longrightarrow \mathbb{P}^{rs+r+s}_Z.$$

As closed immersions are stable under base extension, it suffices to prove this in the special case when $Z = \operatorname{Spec}\mathbb{Z}$. Put coordinates Z_{ij} on $\mathbb{P}^{rs+r+s}_Z$. $\mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z$ is covered by open affine subsets

$$U_i \underset{\mathbb{Z}}{\times} U_j = \operatorname{Spec}\mathbb{Z}[x_0, x_1, \dots, \hat{x}_i, \dots, x_m, y_0, y_1, \dots, \hat{y}_j, \dots, y_n].$$

Define a morphism $U_i \underset{\mathbb{Z}}{\times} U_i \longrightarrow U_{ij}$ on this open subset by the rule

$$(x_0, x_1, \dots, \hat{x}_i, \dots, x_m, y_0, y_1, \dots, \hat{y}_j, \dots, y_n) \longrightarrow (x_k y_l).$$

It is easy to see that these morphisms patch together to give a morphism σ. Let I be the ideal generated by the polynomials

$$Z_{ij}Z_{kl} - Z_{il}Z_{kj}.$$

It is proved in an old hwk set that these equations generate the ideal of the image of σ. In particular the image of σ is closed. To check that σ is a closed immersion it suffices to check that the map of sheaves

$$\mathcal{O}_{\mathbb{P}^{rs+r+s}_Z} \longrightarrow \sigma_* \mathcal{O}_{\mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z},$$

is surjective. As this can be checked locally, we can work on one of the standard open affines. In this case we have already seen that the map on coordinate rings is surjective, so that the map on stalks is certainly surjective.

Now suppose that $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two projective morphisms. By assumption, f and g factor as $i\colon X \longrightarrow \mathbb{P}^r_Y$ and $j\colon Y \longrightarrow \mathbb{P}^s_Z$ followed by the canonical projections, where i and j are closed immersions. Note that there is a commutative diagram (which also happens to be a fibre square; we won't need this)

$$\begin{tikzcd} \mathbb{P}^r_Y \arrow[r] \arrow[d] & \mathbb{P}^r_Z \arrow[d] \\ Y \arrow[r, "g"] & Z, \end{tikzcd}$$

by the functorial properties of $\mathbb{P}^r_S = \mathbb{P}^r_{\mathbb{Z}} \underset{\mathbb{Z}}{\times} S$. It follows that there is a commutative diagram

$$\begin{tikzcd} \mathbb{P}^r_Y \arrow[r] \arrow[d] & Y \arrow[d, "j"'] \\ \mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z \arrow[r] & \mathbb{P}^s_Z. \end{tikzcd}$$

As

$$(\mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z) \underset{\mathbb{P}^s_Z}{\times} Y = \mathbb{P}^r_Z \underset{Z}{\times} Y = \mathbb{P}^r_Y,$$

this diagram is a fibre square.

As j is a closed immersion and closed immersions are stable under base extension, it follows that the morphism

$$\mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z,$$

is a closed immersion. Composing with i, and the Segre morphism $\mathbb{P}^r_Z \underset{Z}{\times} \mathbb{P}^s_Z \longrightarrow \mathbb{P}^{rs+r+s}_Z$ and using the fact that a composition of closed immersions is a closed immersion, we get a closed immersion

$$X \longrightarrow \mathbb{P}^{rs+r+s}_Z,$$

over Z, which is what we had to prove.

4.10. (a) Assume that this result holds if X is irreducible.

As X is of finite type over S and S is Noetherian, X is Noetherian. It follows that X is a finite union of irreducible components $X_1, X_2, \dots, X_k$. As the natural inclusion $X_i \longrightarrow X$ is a closed immersion, a closed immersion is proper and the composition of proper morphisms is proper, it follows that the natural morphism $X_i \longrightarrow S$ is proper. By hypothesis, for each i, we may find a morphism $g_i\colon X'_i \longrightarrow X_i$ and an open subset $U_i \subset X_i$ such that $g_i^{-1}(U_i) \longrightarrow U_i$ is an isomorphism and $X'_i \longrightarrow S$ is projective. Let X' be the disjoint union of the X'_i. Then the natural morphism $X' \longrightarrow S$ is projective.

On the other hand, let X_0 be the disjoint union of $X_1, X_2, \dots, X_k$. Then there are natural morphisms, $X' \longrightarrow X_0$ and $h\colon X_0 \longrightarrow X$. There is an open subset V of X such that $h^{-1}(V) \longrightarrow V$ is an isomorphism. Let U_0 be the union of U_i and let U be the image of $U_0 \cap h^{-1}(V)$. Then U is an open subset of X (indeed it is an open subset of V) and if $g\colon X' \longrightarrow X$ denotes the composition, then $g^{-1}(U) \longrightarrow U$ is an isomorphism.
(b) As S is Noetherian we can cover S by finitely many open affines $S_1, S_2, \dots, S_k$. As f is of finite type we can cover $f^{-1}(S_j = \operatorname{Spec} A)$ by finitely many open affines $U_i = \operatorname{Spec} B \subset X$, $U_1, U_2, \dots, U_n$ where B is a finitely generated A-algebra. If we pick generators $a_1, a_2, \dots, a_m$ for A as a A-algebra then we get a surjective rimg homomorphism

$$A[x_1, x_2, \dots, x_m] \longrightarrow B.$$

Thus there is a closed immersion $U_i \longrightarrow \mathbb{A}^m_{S_j}$. As there is an open immersion $S_i \longrightarrow S$ it follows that there is an open immersion $\mathbb{A}^m_{S_i} \longrightarrow \mathbb{A}^m_S$. Composing with the open immersion $\mathbb{A}^m_S \longrightarrow \mathbb{P}^m_S$, we get an open immersion $\mathbb{A}^m_{S_i} \longrightarrow \mathbb{P}^m_S$. Taking the closure P_i of the image of U_i in $\mathbb{P}^m_S$, we get an open immersion $U_i \longrightarrow P_i$, where P_i is projective over S. In particular $U_1, U_2, \dots, U_n$ are quasi-projective over S.
(c) Note that $h\colon X' \longrightarrow P$ is proper as $P \longrightarrow S$ is separated and the composition $X' \longrightarrow S$ is proper (see (d) of (II.4.8)). In particular h is closed. Given $x' \in X'$ let $x \in X$ be the image. Then $x \in U_i$, some $1 \leq i \leq n$. Let $p \in P$ be the image of x in P and let p_i be the image of p in P_i under the natural projection. By assumption the induced morphism $h_i\colon X' \longrightarrow \mathcal{O}_{P_i}$ is a closed immersion in a neighbourhood of x. Since closed immersions satisfy properties (a-c) of (II.4.8) it follows that h is closed immersion in a neighbourhood of x. Therefore h is a closed immersion.
(d) As the natural morphism $U \longrightarrow U_i$ is an open immersion and the composition of open immersions is an open immersion, it follows that $U \longrightarrow X$ and $U \longrightarrow P_i$ are all open immersions. But then $U \longrightarrow P$ is an isomorphism onto its image and $g^{-1}(U) \longrightarrow U$ is an isomorphism.
1. Suppose that X is contained in a coordinate hyperplane. If this coordinate hyperplane is defined by the equation $X_i = 0$ then X is defined by the monomial X_i and polynomials which don't involve X_i. Replacing $\mathbb{P}^n$ by this coordinate hyperplane and applying induction we may assume therefore assume that X intersects the torus $G = \mathbb{G}^n_m \subset \mathbb{P}^n$. Acting by an element of G won't change whether or not X is defined by binomials. So we may as well assume that the identity $e = [1 : 1 : 1 : \cdots : 1]$ of G is contained in X.
By assumption we may find a dense open subset $H = \mathbb{G}^k_m \subset X$ isomorphic to a torus and a group homomorphism $\rho\colon H \longrightarrow G$. Replacing H

by its image we may assume that $H \subset G$. Let Z be the closure of X. Then H is a dense open subset of Z and the action of H extends to Z. Therefore Z is a non-normal toric variety and the natural inclusion of Z into $\mathbb{P}^n$ is a toric morphism.
Now G acts on $K[X_0, X_1, \dots, X_n]$ and H acts on $K[X_0, X_1, \dots, X_n]$ by restriction. $K[X_0, X_1, \dots, X_n]$ decomposes as a direct sum of eigenspaces and these eigenspaces are direct sums of eigenspaces for the action of G, that is the eigenspaces of the action of H have a basis of monomials. Let I be the ideal of Z. Then I is invariant under the action of H and so I is generated by eigenpolynomials F.
Suppose that we pick two monomials M_1 and M_2 of the same degree with the same eigenvalue. Now $G = M_1 - M_2$ vanishes at e. Therefore it vanishes on the orbit of e, that is on a dense subset of Z. Therefore G vanishes on Z. But then it is clear that every eigenpolynomial $F \in I$ is a sum of binomials $G \in I$.
As both X and Z are unions of orbits of H it follows that $Z - X$ is a union of orbits of H. Since H is a subgroup of G these orbits are contained in the coordinate hyperplanes. Let $\nu \colon Y \longrightarrow Z$ be the normalisation of Z. Then $\nu^{-1}(H) \longrightarrow H$ is an isomorphism and the action of H on Z lifts to an action on Y, by the universal property of the normalisation. Hence Y is a toric variety. By the general theory of toric varieties it follows that Y has finitely many orbits all of which are toric varieties. Therefore Z has finitely many orbits. It follows that the orbits of H on Z are just the intersection of the orbits of G with Z, which are given by the coordinate linear subspaces. Thus $X \subset Z$ is given by the non-vanishing of some mononomials $G_1, G_2, \dots, G_q$.
The last statement about the point $[1 : 1 : 1 : \cdots : 1]$ is immediate.
2. Let U be the free abelian monoid generated by $v_1, v_2, \dots, v_m$ (so that U is abstractly isomorphic to $\mathbb{N}^m$). Define a monoid homomorphism $U \longrightarrow S_\sigma$ by sending v_i to u_i. This is surjective and the kernel is generated by relations of the form

$$\sum a_i v_i - \sum b_i v_i,$$

where

$$\sum a_i u_i = \sum b_i u_i.$$

The group algebra A_σ is generated by $x_i = \chi^{u_i}$. Define a ring homomorphism

$$K[x_1, x_2, \dots, x_n] \longrightarrow A_\sigma,$$

by sending x_i to χ^{u_i}. Then the kernel is generated by equations of the form

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} = x_1^{b_1} x_2^{b_2} \dots x_m^{b_m},$$

where
$$\sum a_i u_i = \sum b_i u_i,$$
since if we quotient out by these relations then we get the vector space spanned by the monomials χ^u, $u \in S_\sigma$.
3. Let Z be the closure of X. Then Z is an irreducible projective variety defined by the vanishing of binomials. We first prove the stronger statement that Z is a non-normal toric variety. As Z is defined by binomials, Hilbert's basis theorem implies that Z is defined by finitely many binomials. If Z is contained in a coordinate hyperplane H then we might as well replace $\mathbb{P}^n$ by this coordinate hyperplane. So we may assume that Z intersects the torus $G = \mathbb{G}_m^n \subset \mathbb{P}^n$. If we act by G this won't change binomial equations, so that we might as well suppose that Z contains the identity $e = [1 : 1 : 1 : \cdots : 1]$ so that the equations defining Z take the form monomial equals monomial.
Let $W \subset \mathbb{A}^{n+1}_K$ be the affine variety defined by the same polynomials as Z. Suppose that W is defined by monomial equations of the form
$$x_0^{a_1} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} = x_0^{b_0} x_1^{b_1} x_2^{b_2} \dots x_n^{b_n}.$$
Let U be the free monoid with generators $v_0, v_1, \dots, v_n$ and let S be the quotient monoid by the relations
$$\sum a_i v_i - \sum b_i v_i.$$
Then the coordinate ring of W is isomorphic to $K[S]$. Embed $U \subset \mathbb{R}^n$. Then the vectors $\sum a_i v_i - \sum b_i v_i$ define a subspace U_0. If we project U onto $U/U_0 = \mathbb{R}^m$ this defines an embedding $S \subset M \simeq \mathbb{Z}^m$. Let $\tau \subset M_{\mathbb{R}}$ be the cone spanned by the images $u_0, u_1, \dots, u_n$ of $v_0, v_1, \dots, v_n$. Then τ is a rational polyhedral cone. Let $\sigma = \check{\tau}$ be the dual cone. We may assume that τ spans $M_{\mathbb{R}}$ so that σ is strongly convex. Then $\tau = \check{\sigma}$ and we have already seen that $S \subset S_\sigma$ is a non-normal affine toric variety defined by the same equations as W. In other words W is a non-normal affine toric variety. But then the action of $\mathbb{G}_m^m$ descends to an action on Z, so that Z is a non-normal toric projective variety and the natural inclusion morphism $Z \longrightarrow \mathbb{P}^n$ is a toric morphism.
As in the proof of (1) it follows that the action of the torus $H \subset Z$ on Z has only finitely many orbits which correspond to the finitely many coordinate linear subspaces. So X is a union of orbits and it follows that X is a toric variety and that the natural inclusion of X into $\mathbb{P}^n$ is a toric morphism.
We saw in class that the curve $C = V(y^2 - x^3) \subset \mathbb{A}^2_K$ admits an action of $\mathbb{G}_m$ and yet this curve is not normal. On the other hand, it is defined

by the vanishing of Y^2Z-X^3 and the non-vanishing of Z, both of which are binomials.