9. Linear systems

Theorem 9.1. *Let X be a scheme over a ring A.*

(1) *If $\phi\colon X \longrightarrow \mathbb{P}^n_A$ is an A-morphism then $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, where $s_i = \phi^* x_i$.*
(2) *If $\mathcal{L}$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, then there is a unique A-morphism $\phi\colon X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ and $s_i = \phi^* x_i$.*

Proof. It is clear that $\mathcal{L}$ is an invertible sheaf. Since $x_0, x_1, \dots, x_n$ generate the ring $A[x_0, x_1, \dots, x_n]$, it follows that $x_0, x_1, \dots, x_n$ generate the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(1)$. Thus $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. Hence (1).

Now suppose that $\mathcal{L}$ is an invertible sheaf generated by $s_0, s_1, \dots, s_n$. Let

$$X_i = \{\, p \in X \mid s_i \notin \mathfrak{m}_p \mathcal{L}_p \,\}.$$

Then X_i is an open subset of X and the sets $X_0, X_1, \dots, X_n$ cover X. Define a morphism

$$\phi_i\colon X_i \longrightarrow U_i,$$

where U_i is the standard open subset of $\mathbb{P}^n_A$, as follows: Since

$$U_i = \operatorname{Spec} A[y_0, y_1, \dots, y_n],$$

where $y_j = x_j/x_i$, is affine, it suffices to give a ring homomorphism

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}).$$

We send y_j to s_j/s_i, and extend by linearity. The key observation is that the ratio is a well-defined element of $\mathcal{O}_{X_i}$, which does not depend on the choice of isomorphism $\mathcal{L}|_{X_i} \simeq \mathcal{O}_{X_i}$.

It is then straightforward to check that the set of morphisms $\{\phi_i\}$ glues to a morphism ϕ with the given properties. $\square$

Example 9.2. *Let $X = \mathbb{P}^1_k$, $A = k$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(2)$.*

In this case, global sections of $\mathcal{L}$ are generated by S^2, ST and T^2. This morphism is represented globally by

$$[S : T] \longrightarrow [S^2 : ST : T^2].$$

The image is the conic $XZ = Y^2$ inside $\mathbb{P}^2_k$.

More generally one can map $\mathbb{P}^1_k$ into $\mathbb{P}^n_k$ by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(n)$. More generally still, one can map $\mathbb{P}^m_k$ into $\mathbb{P}^n_k$ using the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(1)$.

Corollary 9.3.

$$\operatorname{Aut}(\mathbb{P}^n_k) \simeq \operatorname{PGL}(n+1, k).$$

Proof. First note that $\mathrm{PGL}(n+1,k)$ acts naturally on $\mathbb{P}^n_k$ and that this action is faithful.

Now suppose that $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$. Let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. Since $\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, it follows that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(\pm 1)$. As $\mathcal{L}$ is globally generated, we must have $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. Let $s_i = \phi^* x_i$. Then $s_0, s_1, \dots, s_n$ is a basis for the k-vector space $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. But then there is a matrix

$$A = (a_{ij}) \in \mathrm{GL}(n+1,k) \qquad \text{such that} \qquad s_i = \sum_{ij} a_{ij} x_j.$$

Since the morphism ϕ is determined by $s_0, s_1, \dots, s_n$, it follows that ϕ is determined by the class of A in $\mathrm{GL}(n+1,k)$. $\square$

Lemma 9.4. *Let $\phi \colon X \longrightarrow \mathbb{P}^n_A$ be an A-morphism. Then ϕ is a closed immersion if and only if*

(1) *$X_i = X_{s_i}$ is affine, and*
(2) *the natural map of rings*

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \qquad \textit{which sends} \qquad y_i \longrightarrow \frac{\sigma_i}{\sigma_j},$$

is surjective.

Proof. Suppose that ϕ is a closed immersion. Then X_i is isomorphic to $\phi(X) \cap U_i$, a closed subscheme of affine space. Thus X_i is affine. Hence (1) and (2) follows as we have surjectivity on all of the localisations.

Now suppose that (1) and (2) hold. Then X_i is a closed subscheme of U_i and so X is a closed subscheme of $\mathbb{P}^n_A$. $\square$

Theorem 9.5. *Let X be a projective scheme over an algebraically closed field k and let $\phi \colon X \longrightarrow \mathbb{P}^n_k$ be a morphism over k, which is given by an invertible sheaf $\mathcal{L}$ and global sections $s_0, s_1, \dots, s_n$ which generate $\mathcal{L}$. Let $V \subset \Gamma(X, \mathcal{L})$ be the space spanned by the sections.*

Then ϕ is a closed immersion if and only if

(1) *V **separates points**: that is, given p and $q \in X$ there is $\sigma \in V$ such that $\sigma \in \mathfrak{m}_P \mathcal{L}_p$ but $\sigma \notin \mathfrak{m}_q \mathcal{L}_q$.*
(2) *V **separates tangent vectors**: that is, given $p \in X$ the set*

$$\{\, \sigma \in V \mid \sigma \in \mathfrak{m}_p \mathcal{L}_p \,\},$$

spans $\mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p$.

Proof. Suppose that ϕ is a closed immersion. Then we might as well consider $X \subset \mathbb{P}^n_k$ as a closed subscheme. In this case (1) is clear. Just pick a linear function on the whole of $\mathbb{P}^n_k$ which vanishes at p but not at q (equivalently pick a hyperplane which contains p but not q).

Similarly linear functions on $\mathbb{P}^n_k$ separate tangent vectors on the whole of projective space, so they certainly separate on X.

Now suppose that (1) and (2) hold. Then ϕ is clearly injective. Since X is proper over $\operatorname{Spec} k$ and $\mathbb{P}^n_k$ is separated over $\operatorname{Spec} k$ it follows that ϕ is proper. In particular $\phi(X)$ and ϕ is a homeomorphism onto $\phi(X)$. It remains to show that the map on stalks

$$\mathcal{O}_{\mathbb{P}^n_k,p} \longrightarrow \mathcal{O}_{X,x},$$

is surjective. But the same piece of commutative algebra as we used in the proof of the inverse function theorem, works here. □

Definition 9.6. *Let X be a noetherian scheme. We say that an invertible sheaf $\mathcal{L}$ is **ample** if for every coherent sheaf $\mathcal{F}$ there is an integer $n_0 > 0$ such that $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^n$ is globally generated, for all $n \geq n_0$.*

Lemma 9.7. *Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme. TFAE*

(1) *$\mathcal{L}$ is ample.*
(2) *$\mathcal{L}^m$ is ample for all $m > 0$.*
(3) *$\mathcal{L}^m$ is ample for some $m > 0$.*

Proof. (1) implies (2) implies (3) is clear.

So assume that $\mathcal{M} = \mathcal{L}^m$ is ample and let $\mathcal{F}$ be a coherent sheaf. For each $0 \leq i \leq m-1$, let $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{L}^i$. By assumption there is an integer n_i such that $\mathcal{F}_i \otimes \mathcal{M}^n$ is globally generated for all $n \geq n_i$. Let n_0 be the maximum of the n_i. If $n \geq (n_0+1)m$, then we may write $n = qm + i$, where $0 \leq i \leq m-1$ and $q \geq n_0 \geq n_i$.

But then

$$\mathcal{F} \otimes \mathcal{L}^m = \mathcal{F}_i \otimes \mathcal{M}^q,$$

which is globally generated. □

Theorem 9.8. *Let X be a scheme of finite type over a Noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X.*

Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample for some $m > 0$.

Proof. Suppose that $\mathcal{L}^m$ is very ample. Then there is an immersion $X \subset \mathbb{P}^r_A$, for some positive integer r, and $\mathcal{L}^m = \mathcal{O}_X(1)$. Let $\bar{X}$ be the closure. If $\mathcal{F}$ is any coherent sheaf on X then there is a coherent sheaf $\overline{\mathcal{F}}$ on $\bar{X}$, such that $\mathcal{F} = \overline{\mathcal{F}}|_X$. By Serre's result, $\overline{\mathcal{F}}(k)$ is globally generated for some positive integer k. It follows that $\mathcal{F}(k)$ is globally generated, so that $\mathcal{L}^m$ is ample, and the result follows by (9.7).

Conversely, suppose that $\mathcal{L}$ is ample. Given $p \in X$, pick an open affine neighbourhood U of p so that $\mathcal{L}|_U$ is free. Let $Y = X - U$, give it the reduced induced strucure, with ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent.

Pick $n > 0$ so that $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated. Then we may find $s \in \mathcal{I} \otimes \mathcal{L}^n$ not vanishing at p. We may identify s with $s' \in \mathcal{O}_U$ and then $p \in U_s \subset U$, an affine subset of X.

By compactness, we may cover X by such open affines and we may assume that n is fixed. Replacing $\mathcal{L}$ by $\mathcal{L}^n$ we may assume that $n = 1$. Then there are global sections $s_1, s_2, \dots, s_k \in H^0(X, \mathcal{L})$ such that $U_i = U_{s_i}$ is an open affine cover.

Since X is of finite type, each $B_i = H^0(U_i, \mathcal{O}_{U_i})$ is a finitely generated A-algebra. Pick generators b_{ij}. Then $s^n b_{ij}$ lifts to $s_{ij} \in H^0(X, \mathcal{L}^n)$. Again we might as well assume $n = 1$.

Now let $\mathbb{P}^N_A$ be the projective space with coordinates $x_1, x_2, \dots, x_k$ and x_{ij}. Locally we can define a map on each U_i to the standard open affine, by the obvious rule, and it is standard to check that this glues to an immersion. □

Definition 9.9. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety over an algebraically closed field. Let $s \in H^0(X, \mathcal{L})$. The divisor (s) of zeroes of s is defined as follows. By assumption we may cover X by open subsets U_i over which we may identify $s|_{U_i}$ with $f_i \in \mathcal{O}_{U_i}$. The defines a Cartier divisor $\{(U_i, f_i)\}$.*

It is a simple matter to check that the Cartier divisor does not depend on our choice of trivialisations. Note that as X is smooth the Cartier divisor may safely be identified with the corresponding Weil divisor.

Lemma 9.10. *Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor and let $\mathcal{L} = \mathcal{O}_X(D_0)$.*

1. *If $s \in H^0(X, \mathcal{L})$, $s \neq 0$ then $(s) \sim D_0$.*
2. *If $D \geq 0$ and $D \sim D_0$ then there is a global section $s \in H^0(X, \mathcal{L})$ such that $D = (s)$.*
3. *If $s_i \in H^0(X, \mathcal{L})$, $i = 1$ and 2, are two global sections then $(s_1) = (s_2)$ if and only if $s_2 = \lambda s_1$ where $\lambda \in k^*$.*

Proof. As $\mathcal{O}_X(D_0) \subset \mathcal{K}$, the section s corresponds to a rational function f. If D_0 is the Cartier divisor $\{(U_i, f_i)\}$ then $\mathcal{O}_X(D_0)$ is locally generated by f_i^{-1} so that multiplication by f_i induces an isomorphism with $\mathcal{O}_{U_i}$. D is then locally defined by ff_i. But then

$$D = D_0 + (f).$$

Hence (1).

Now suppose that $D > 0$ and $D = D_0 + (f)$. Then $(f) \geq -D_0$. Hence

$$f \in H^0(X, \mathcal{O}_X(D_0)) \subset H^0(X, \mathcal{K}) = K(X),$$

and the divisor of zeroes of f is D. This is (2).

Now suppose that $(s_1) = (s_2)$. Then

$$D_0 + (f_1) = (s_1) = (s_2) = D_0 + (f_2).$$

Cancelling, we get that $(f_1) = (f_2)$ and the rational function f_1/f_2 has no zeroes nor poles. Since X is a projective variety, $f_1/f_2 = \lambda$, a constant. $\square$

Definition 9.11. *Let D_0 be a divisor. The* ***complete linear system*** *associated to D_0 is the set*

$$|D_0| = \{\, D \in \operatorname{Div}(X) \,|\, D \geq 0, D \sim D_0 \,\}.$$

We have seen that

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Thus $|D|$ is naturally a projective space.

Definition 9.12. *A* ***linear system*** *is any linear subspace of a complete linear system $|D_0|$.*

In other words, a linear system corresponds to a linear subspace, $V \subset H^0(X, \mathcal{O}_X(D_0))$. We will then write

$$|V| = \{\, D \in |D_0| \,|\, D = (s), s \in V \,\} \simeq \mathbb{P}(V) \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Definition 9.13. *Let $|V|$ be a linear system. The* ***base locus*** *of $|V|$ is the intersection of the elements of $|V|$.*

Lemma 9.14. *Let X be a smooth projective variety over an algebraically closed field, and let $|V| \subset |D_0|$ be a linear system.*

V generates $\mathcal{O}_X(D_0)$ if and only if $|V|$ is base point free.

Proof. If V generates $\mathcal{O}_X(D_0)$ then for every point $x \in X$ we may find an element $\sigma \in V$ such that $\sigma(x) \neq 0$. But then $D = (\sigma)$ does not contain x, and so the base locus is empty.

Conversely suppose that the base locus is empty. The locus where V does not generated $\mathcal{O}_X(D_0)$ is a closed subset Z of X. Pick $x \in Z$ a closed point. By assumption we may find $D \in |V|$ such that $x \notin D$. But then if $D = (\sigma)$, $\sigma(x) \neq 0$ and σ generates the stalk $\mathcal{L}_x$, a contradiction. Thus Z is empty and $\mathcal{O}_X(D_0)$ is globally generated. $\square$

Example 9.15. *Consider $\mathcal{O}_{\mathbb{P}^1}(4)$. The complete linear system $|4p|$ defines a morphism into $\mathbb{P}^4$, where $p = [0 : 1]$ and $q = [1 : 0]$, given by $\mathbb{P}^1 \longrightarrow \mathbb{P}^4$, $[S : T] \longrightarrow [S^4 : ST^3 : S^2T^2 : ST^3 : T^4]$. If we project from $[0 : 0 : 1 : 0 : 0]$ we will get a morphism into $\mathbb{P}^3$, $[S : T] \longrightarrow [S^4 : ST^3 : ST^3 : T^4]$. This corresponds to the sublinear system spanned by $4p$, $3p + q$, $p + 3q$, $4q$.*

Consider $\mathcal{O}_{\mathbb{P}^2}(2)$ and the corresponding complete linear system. The map associated to this linear system is the Veronese embedding $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$, $[X : Y : Z] \longrightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$.

Note also the notion of separating points and tangent directions becomes a little clearer in this more geometric setting. Separating points means that given x and $y \in X$, we can find $D \in |V|$ such that $x \in D$ and $y \notin D$. Separating tangent vectors means that given any irreducible length two zero dimensional scheme z, with support x, we can find $D \in |V|$ such that $x \in D$ but z is not contained in D. In fact the condition about separating tangent vectors is really the limiting case of separating points.

Thinking in terms of linear systems also presents an inductive approach to proving global generation. Suppose that we consider the complete linear system $|D|$. Suppose that we can find $Y \in |D|$. Then the base locus of $|D|$ is supported on Y. On the other hand suppose that $\mathcal{I}$ is the ideal sheaf of Y in X. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

As X is smooth D is Cartier and $\mathcal{O}_X(D)$ is an invertible sheaf. Tensoring by locally free preserves exactness, so there are short exact sequences,

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_Y(mD) \longrightarrow 0.$$

Taking global sections, we get

$$0 \longrightarrow H^0(X, \mathcal{I}(mD) \longrightarrow H^0(X, \mathcal{O}_X(mD) \longrightarrow H^0(Y, \mathcal{O}_Y(mD).$$

At the level of linear systems there is therefore a linear map

$$|D| \longrightarrow |D|_Y|.$$

It is interesting to see what happens for toric varieties. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_{\mathbb{R}}$. Recall that we can associate to a T-Cartier divisor $D = \sum a_i D_i$, a continuous piecewise linear function

$$\phi_D \colon |F| \longrightarrow \mathbb{R},$$

where $|F| \subset N_{\mathbb{R}}$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to D a rational polyhedron

$$\begin{aligned} P_D &= \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall i \,\} \\ &= \{\, u \in M_{\mathbb{R}} \,|\, u \geq \phi_D \,\}. \end{aligned}$$

Lemma 9.16. *If X is a toric variety and D is T-Cartier then*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Proof. Suppose that $\sigma \in F$ is a cone. Then, we have already seen that

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall v_i \in \sigma \,\}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \cap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \cap_{\sigma \in F} P_D(\sigma),$$

the result is clear. $\square$

It is interesting to compute some examples. First, suppose we consider $\mathbb{P}^1$. A T-Cartier divisor is a sum $ap + bq$ (p and q fixed points). The corresponding function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

The corresponding polytope is the interval

$$[-a, b] \subset \mathbb{R} = N_{\mathbb{R}}.$$

There are $a+b+1$ integral points, corresponding to the fact that there are $a+b+1$ monomials of degree $a+b$. For $\mathbb{P}^2$ and dD_3, P_D is the convex hull of $(0,0)$, $(d,0)$ and $(0,d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula.

Let D be a Cartier divisor on a toric variety $X = X(F)$ given by a fan F. It is interesting to consider when the complete linear system $|D|$ is base point free. Since any Cartier divisor is linearly equivalent to a T-Cartier divisor, we might as well suppose that $D = \sum a_i D_i$ is T-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone $\sigma \in F$ the point $x_\sigma \in U_\sigma$ is not in the base locus. It is also clear that if x_σ is not in the base locus of $|D|$ then in fact one can find a

T-Cartier divisor $D' \in |D|$ which does not contain x_σ. Equivalently we can find $u \in M$ such that

$$\langle u, v_i \rangle \geq -a_i,$$

with strict equality if $v_i \in \sigma$. The interesting thing is that we can reinterpret this condition using ϕ_D.

Definition 9.17. *The function $\phi\colon V \longrightarrow \mathbb{R}$ is **upper convex** if*

$$\phi(\lambda v + (1-\lambda)w) \geq \lambda\phi(v) + (1-\lambda)w \qquad \forall v, w \in V.$$

*When we have a fan F and ϕ is linear on each cone σ, then ϕ is called **strictly upper convex** if the linear functions $u(\sigma)$ and $u(\sigma')$ are different, for different maximal cones σ and σ'.*

Theorem 9.18. *Let $X = X(F)$ be the toric variety associated to a T-Cartier divisor D.*

Then

1. *$|D|$ is base point free if and only if ψ_D is upper convex.*
2. *D is very ample if and only if ψ_D is strictly upper convex and the semigroup S_σ is generated by*

$$\{\, u - u(\sigma) \,|\, u \in P_D \cap M \,\}.$$

Proof. (1) follows from the remarks above. (2) is proved in Fulton's book. □

For example if $X = \mathbb{P}^1$ and

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

so that $D = ap + bq$ then ϕ is upper convex if and only if $a + b \geq 0$ in which case D is base point free. D is very ample if and only if $a+b > 0$. When ϕ is continuous and linear on each cone σ, we may restate the upper convex as saying that the graph of ϕ lies under the graph of $u(\sigma)$. It is strictly upper convex if it lies strictly under the graph of $u(\sigma)$, for all n-dimensional cones σ.

Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset N_{\mathbb{R}} = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect v_1 to v_5, v_3 to v_7 and v_2 to v_6 and v_5 to v_6, v_6 to v_7 and v_7 to v_5.

It is not hard to check that X is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $N_{\mathbb{R}}$).

Suppose that ψ is strictly upper convex. Let w be the midpoint of the line connecting v_1 and v_5. Then

$$w = \frac{v_1 + v_5}{2} = \frac{v_3 + v_6}{2}.$$

Since v_1 and v_5 belong to the same maximal cone, ψ is linear on the line connecting them. In particular

$$\psi(w) = \psi(\frac{v_1 + v_5}{2}) = \frac{1}{2}\psi(v_1) + \frac{1}{2}\psi(v_5).$$

Since v_1, v_5 and v_3 belong to the same cone and v_6 does not, by strict convexity,

$$\psi(w) = \psi(\frac{v_3 + v_6}{2}) > \frac{1}{2}\psi(v_3) + \frac{1}{2}\psi(v_6).$$

Putting all of this together, we get

$$\psi(v_1) + \psi(v_5) > \psi(v_2) + \psi(v_6).$$

By symmetry

$$\begin{aligned} \psi(v_1) + \psi(v_5) &> \psi(v_3) + \psi(v_6) \\ \psi(v_2) + \psi(v_6) &> \psi(v_1) + \psi(v_7) \\ \psi(v_3) + \psi(v_7) &> \psi(v_2) + \psi(v_5). \end{aligned}$$

But adding up these three inequalities gives a contradiction.

Consider another application of the ideas behind this section. Consider the problem of parametrising subvarieties or subschemes X of projective space $\mathbb{P}^r_k$. Any subscheme is determined by the homogeneous ideal $I(X)$ of polynomials vanishing on X. As in the case of zero dimensional schemes, we would like to reduce to the data of a vector subspace of fixed dimension in a fixed vector space. The obvious thing to consider is polynomials of degree d and the vector subspace of polynomials of polynomials of degree d vanishing on X. But how large should we take d to be?

The first observation is that if $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^r_k$ then

$$I_d = H^0(\mathbb{P}^r_k, \mathcal{I}(d)),$$

where $\mathcal{I}(d)$ is the Serre twist. To say that I_d determines X, is essentially equivalent to saying that $\mathcal{I}(d)$ is globally generated. Fixing some data about X (in the case of zero dimensional schemes this would be the length) we would then like a positive integer d_0 such that if $d \geq d_0$ then two things are true:

- $\mathcal{I}(d)$ is globally generated.
- $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$, the dimension of the space of global sections, is independent of X.

Now there is a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^r_k} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting by d, we get

$$0 \longrightarrow \mathcal{I}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^r_k}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

Taking global sections gives another exact sequence.

$$0 \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{I}(d)) \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d)).$$

Again, it would be really nice if this exact sequence were exact on the right. Then global generation of $\mathcal{I}(d)$ would be reduced to global generation of $\mathcal{O}_X(d)$ and one could read of $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$ from $h^0(X, \mathcal{O}_X(d))$.