

6. NON-REDUCED SCHEMES AND FLAT LIMITS OF ZERO  
DIMENSIONAL SCHEMES

**6.1. Some examples of zero dimensional schemes.** We would like to understand the geometric content behind non-reduced schemes. Let us start with a simple example. Let  $k$  be a field and let

$$A = \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

Consider  $T = \text{Spec } A$ . Clearly  $A$  contains only one prime ideal, namely  $\langle \epsilon \rangle$ . Thus  $T$  has only one point. However the stalk of the structure sheaf is not a field. To get a picture of  $T$ , we can embed this scheme in  $\mathbb{A}_k^1$ ,

$$k[x] \longrightarrow A \quad \text{to get} \quad T \longrightarrow \mathbb{A}_k^1,$$

where  $x \longrightarrow \epsilon$ . In fact we can think of two points in  $\mathbb{A}_k^1$ ,  $p_t$  and  $q_t$  and think about what happens when  $p_t$  approaches  $q_t$ . We might as well suppose that  $q_t$  is fixed,  $q_t = \langle x \rangle$ . Let  $p_t = \langle x - t \rangle$ . The ideal of the union is then

$$\langle x(x - t) \rangle.$$

As  $t$  approaches 0, it is natural to identify the limit as

$$\langle x^2 \rangle,$$

which is the ideal of  $T \subset \mathbb{A}_k^1$ . With this picture, it is natural to think of  $T$  as being a point, together with a tangent direction. Abstractly,  $T$  is a point together with a disembodied tangent vector. Note that this is a very natural way to think of tangent vectors algebraically; if we want to differentiate, then we want to expand in powers of  $\epsilon$  and ignore all terms of degree two and higher. In fact

**Definition 6.1.** *Let  $x \in X$  be a point of a scheme, with residue field  $k$ . The **Zariski tangent space**  $T_x X$  to  $X$  at  $x$  is the  $k$ -vector space of all morphisms over  $\text{Spec } k$ ,*

$$T_x X = \text{Hom}(\text{Spec } k[\epsilon]/\langle \epsilon^2 \rangle, X),$$

where the image of the unique point of  $\text{Spec } k[\epsilon]/\langle \epsilon^2 \rangle$  is  $x$ .

Note that  $T$  has many embeddings into  $\mathbb{A}_k^2$ . Indeed, think of two points approaching each other. If one of the points is fixed to be the origin and the other approaches along a smooth curve, then the limiting subscheme (we will make the naive notion of the limit more formal very shortly) is a copy of  $T$ . However, the two points remember the tangent

direction of their approach. Thus the set of all embeddings of  $T$  into  $\mathbb{A}^2$ , is given by

$$\langle ax + by \rangle + \mathfrak{m}^2,$$

where  $\mathfrak{m}$  is the maximal ideal.

**Definition 6.2.** *Let  $X$  be a zero dimensional scheme over a field  $k$ . The **length** of  $X$  is simply the dimension of the  $k$ -vector space  $\mathcal{O}_X(X)$ .*

Here is another, more direct way, to classify all zero dimensional subschemes of  $\mathbb{A}^2$  of length two supported at the origin. Any such scheme is a closed subscheme. It follows that it is given by a surjective morphism,

$$k[x, y] \longrightarrow k[\epsilon]/\langle \epsilon^2 \rangle$$

Let  $\mathfrak{a}$  be the kernel of this map. Then the radical of  $\mathfrak{a}$  must be the ideal  $m = \langle x, y \rangle$ . Since the square of any element of the image is zero, it follows that

$$m^2 = \langle x^2, xy, y^2 \rangle,$$

must be contained in the kernel. In other words we have the inclusion of subschemes

$$T \subset Z \subset \mathbb{A}_k^2 \quad \text{where} \quad Z = \text{Spec } k[x, y]/\langle x^2, xy, y^2 \rangle.$$

Now

$$k[x, y]/\langle x^2, xy, y^2 \rangle,$$

is a vector space of dimension three over  $k$  and

$$k[\epsilon]/\langle \epsilon^2 \rangle,$$

has dimension two. It follows that there must be a linear form in the kernel, say  $ax + by$ . The quotient then has the right dimension, and this gives us the classification.

Another way to think of this is as follows. Let  $f \in k[x, y] = \mathcal{O}_{\mathbb{A}_k^2}(\mathbb{A}_k^2)$ . Then we may restrict  $f$  to  $T$ . Suppose that  $T$  is given by  $\langle y, x^2 \rangle$ . Let  $g$  be the restriction of  $f$  to  $k[x, y]/\langle y, x^2 \rangle$ . Then  $g$  picks out both the value of  $f$  at the origin and the coefficient of the  $x$ -term of the Taylor series of  $f$ . In other words, the restriction is determined by

$$f(0, 0) \quad \text{and} \quad \left. \frac{\partial f}{\partial x} \right|_{(0,0)}.$$

If we again think of a family of two distinct points approaching each other, then instead of evaluating  $f$  at the two distinct points, we evaluate  $f$  at the point and in the tangent direction of their approach.

Note yet another way to think of an embedding of  $T$  into a smooth variety  $X$ . Let  $C$  be a smooth curve, and let  $x \in C$  be a point of  $C$ . Then if we truncate  $C$  to order two, then we get a copy of  $T \subset X$ .

In other words, we look at the subscheme defined by  $I_C + \mathfrak{m}^2 \subset A$ , where  $U = \text{Spec } A$  is an open affine neighbourhood of  $x$ ,  $I_C$  is the ideal of  $C$  and  $\mathfrak{m}$  is the ideal corresponding to  $x$ . In other words, a closed embedding of  $T$  inside  $X$  is an equivalence class of smooth curves, just like tangent vectors in classical differential geometry.

Consider the intersection of a line with a conic. If the line is not tangent to the conic, then the line intersects the conic in two points and these points span the line. But if the line is tangent to the conic we only get one point. Nevertheless, as a scheme we get a double point. Moreover, we can recover the line from the scheme as the smallest linear space which contains the scheme (aka the span).

Armed with this example, the geometric meaning of other non-reduced schemes becomes a little more clear. For example, consider the scheme

$$Z = \text{Spec } \frac{k[\epsilon]}{\langle \epsilon^3 \rangle}.$$

Consider how to embed  $Z$  inside  $\mathbb{A}_k^2$ . Perhaps the easiest way to proceed is to pick a smooth curve and truncate to order three, rather than two. If the curve is  $y = x^2$ , then we look at

$$\langle y - x^2, x^3 \rangle.$$

This is obviously different from looking at

$$\langle y, x^3 \rangle.$$

Here, we have a tangent direction together with a second order tangent direction. We can think of this scheme as the limit as three points coming together. However in this case, the points need to approach each other along the same smooth curve. Indeed, note that there is in fact another irreducible zero dimensional scheme, up to isomorphism. Consider

$$\text{Spec } k[x, y] / \langle x^2, xy, y^2 \rangle.$$

Note that this contains all length two subschemes of the origin with the same support. In other words, if we look at a function, then its restriction is determined by its value at the origin and two independent derivatives. It is not hard to see that these are the only two possibilities for length three schemes of irreducible schemes.

**6.2. Flat limits.** It is interesting to see how one gets these examples as limits. First suppose that three points approach each other. Fixing one as the origin, we suppose that the other two approach along smooth curves. For example, suppose the points are

$$(0, 0) \quad (t, 0) \quad \text{and} \quad (0, t).$$

The corresponding ideals are

$$\langle x, y \rangle \quad \langle x - t, y \rangle \quad \text{and} \quad \langle x, y - t \rangle.$$

The product of these ideals is then

$$\langle x^3 - x^2t, x^2t + xyt - xt^2, x^2y, xyt, x^2y - xyt, xyt + y^2t - yt^2, xy^2, y^3 - y^2t \rangle,$$

which is the ideal of the union. Consider the limit, as  $t$  goes to zero. If we naively set  $t = 0$  then we get

$$\langle x^3, x^2y, x^2y, xy^2, y^3 \rangle = \mathfrak{m}^3.$$

But this cannot be the limit since the corresponding scheme has length 6 and the limit should be a scheme of length 3. However, for  $t \neq 0$ , we always get  $xy$  in the ideal. Thus the limit must also contain  $xy$ . From there it is easy to see that the limit must also contain  $x^2$  and  $y^2$ , so that we get,

$$\langle x^2, xy, y^2 \rangle = \mathfrak{m}^2,$$

which does indeed correspond to a scheme of length 3. So the flat limit of the three points is the scheme corresponding to  $\mathfrak{m}^2$ .

Now consider the case when we have three points approaching each other along a smooth curve. For example, take the three points

$$(0, 0) \quad (t, t^2) \quad \text{and} \quad (-t, t^2).$$

Since these points lie in the smooth curve  $y - x^2$ , which is isomorphic to  $\mathbb{A}^1$ , it follows that the flat limit is the unique length three scheme supported at the origin and contained in this curve, namely

$$\langle y - x^2, x^3 \rangle.$$

**Definition 6.3.** *We say that a zero dimensional scheme  $z$  is **curvilinear** if it can be embedded in a smooth curve  $C$ .*

Note that a zero dimensional scheme is curvilinear if and only if its irreducible components are curvilinear. On the other hand if  $z$  is irreducible then it is curvilinear if and only if it is isomorphic to

$$\text{Spec} \frac{k[\epsilon]}{\langle \epsilon^l \rangle},$$

for some positive integer  $l$ .

**6.3. Punctual Hilbert scheme.** It is interesting to look at this from a different perspective. One way to think about taking limits, is to think of all our schemes as defining points of a Grassmannian. Indeed, all of our schemes are defined by ideals, such that the quotient is of finite dimension. Now suppose that the support is a fixed point. Then it is not hard to see that if one fixes the length, then some fixed power of the maximal ideal, will be contained in our ideal (in fact the length itself will do). Taking quotients, we get a subvector space of a fixed vector space. Thus the locus of all length  $l$  zero dimensional subschemes is naturally a subset of a Grassmannian. In fact this locus is algebraic. A one parameter family of ideals is then the same as a curve in the Grassmannian. As the Grassmannian is projective the closure of this curve is projective and this defines the flat limit.

The resulting locus is called the punctual Hilbert scheme, and is denoted  $\mathcal{H}_0^l$ . As the name might suggest, this subset of the Grassmannian corresponding to ideals of length  $l$  is not only closed, but it actually inherits the structure of a closed subscheme. The locus of curvilinear schemes is easily seen to be an open subscheme, and it is denoted  $\mathcal{C}_0^l$ .

Let

$$R = \frac{k[x, y]}{\mathfrak{m}^l}.$$

Then  $R$  is a finite dimensional vector space over  $k$  and ideals  $I$  in  $k[x, y]$  such that

$$\frac{k[x, y]}{I},$$

has length  $l$  are the same as ideals  $J$  in  $R$  such that the quotient

$$\frac{R}{J},$$

has length  $l$ . In particular  $[J] \in \mathbb{G}(M, N)$  for appropriate (and easily computable) integers  $M$  and  $N$  (indeed,  $M = N - l$ ). Since  $x$  and  $y$  generate the ring  $R$  it follows that  $J$  is an ideal if and only if

$$xJ \subset J \quad \text{and} \quad yJ \subset J.$$

These give equations defining the punctual Hilbert scheme as a closed subscheme of the Grassmannian.

It is interesting to see what happens when we consider all length three schemes supported at a point. In this case, every scheme is certainly a subscheme of

$$\text{Spec } k[x, y]/\mathfrak{m}^3.$$

Now

$$k[x, y]/\mathfrak{m}^3,$$

is a vector space of dimension six, and so we are looking at the Grassmannian of three planes in a vector space of dimension six. The scheme corresponding to  $\mathfrak{m}^2$  is one point in this space. Given any other length three scheme, there is a unique length two subscheme contained in this scheme. It is obtained by truncating the ideal in the obvious way. Now the space of length two schemes is nothing more than the space of all tangent directions at the point, so that this space is isomorphic to  $\mathbb{P}^1$ . Consider the fibre over a point of  $\mathcal{C}_0^2$ . We are looking at all ideals of the form

$$I = \langle y + q \rangle + \mathfrak{m}^3,$$

where  $q$  is a quadratic form. Now  $y(y + q) = y^2 + yq \in I$ . As  $yq \in \mathfrak{m}^3$  it follows that  $y^2 \in I$ . Similarly  $xy \in I$ . But then we may choose  $q = ax^2$ . It is not hard to check that

$$\langle y + ax^2 \rangle + \mathfrak{m}^3 = \langle y + bx^2 \rangle + \mathfrak{m}^3,$$

iff  $a = b$ . Thus there is a morphism from the space of curvilinear schemes of length three down to the space of length two schemes. The fibres of this morphism are  $\mathbb{A}^1$  and the base is  $\mathbb{P}^1$ . Now the unique non-curvilinear scheme is a limit of curvilinear schemes with fixed tangent direction;

$$\lim_{t \rightarrow 0} \langle ty + x^2 \rangle + \mathfrak{m}^3 = \mathfrak{m}^2.$$

Thus the unique non-curvilinear schemes is in the closure of every fibre. It follows that this space is a cone over  $\mathbb{P}^1$ . In fact it is the usual quadric cone in  $\mathbb{A}^3$ .

Note that  $\mathcal{C}_0^3$  is an open subset of the projective scheme  $\mathcal{H}_0^3$ , and more generally, for any  $l$ , the Hilbert scheme gives a projective compactification of the curvilinear locus. It is also interesting to consider different compactifications of the curvilinear locus. For example, we have already seen that there is a closed embedding,

$$\mathcal{C}_0^3 \subset \mathcal{C}_0^3 \times \mathcal{C}_0^2,$$

where we send

$$z \longrightarrow (z, z_2),$$

and  $z_2$  is the unique length two subscheme contained in  $z$  (here we work with irreducible schemes). The closure of the image inside  $\mathcal{H}_0^3 \times \mathcal{H}_0^2$  defines another compactification  $\mathcal{B}^3$  of  $\mathcal{C}_0^3$ . In fact this compactification is smooth; there is an obvious morphism  $\mathcal{B}^3 \longrightarrow \mathcal{H}_0^3$  which just forgets the length two scheme. This morphism is an isomorphism over the curvilinear locus and over the point corresponding to the unique length three scheme which is not curvilinear, we get a whole copy of  $\mathbb{P}^1$ , as any length two scheme is contained in this length three scheme.

**6.4. Hilbert scheme.** In some ways it is more natural to consider all zero dimensional schemes inside  $\mathbb{P}^n$  and not just the punctual ones. This is called the Hilbert scheme and is denoted  $\mathcal{H}^l$ . The construction given above does not work, since there is no obvious fixed finite dimensional vector space in which to work. We will see later that one can construct the full Hilbert scheme using a similar but more sophisticated argument. Assuming the existence of the Hilbert scheme, we can ask what does it look like. Once again the curvilinear locus  $\mathcal{C}^l$  is an open subset.

It is also an interesting question to ask which zero dimensional schemes are limits of curvilinear schemes.

**Theorem 6.4.** *Let  $S$  be a smooth surface. Fix a positive integer  $l$ .*

*Then the Hilbert scheme of zero dimensional schemes of length  $l$  is irreducible and smooth.*

One obvious component of the Hilbert scheme is the closure of the space of curvilinear schemes, which is obviously irreducible (an open subset is simply the product of the surface with itself  $l$  times, minus the diagonals). Thus (6.4) really answers our question for surfaces; every zero dimensional scheme is a limit of curvilinear schemes.

It is interesting to look at the symmetric product. One way to compactify the space of  $l$  distinct unordered points is to consider

$$S^{(l)} = S^l / \Sigma_l$$

where  $S^l$  is the  $l$ -fold product of  $S$  with itself and  $\Sigma_l$  is the symmetric group, acting on the obvious way on  $S^l$ . It turns out that there is a natural morphism

$$\mathcal{H}^l \longrightarrow S^{(l)},$$

which just assigns to a scheme its support (in fact the space on the right is the Chow scheme). This map is birational and in fact the Hilbert scheme gives a desingularisation of the symmetric product, which is in fact highly singular.

Perhaps surprisingly most of this fails in higher dimensions.

**6.5. Non-reduced curves and embedded components.** We can also look at doubled curves. For example, consider

$$\langle x^2 \rangle \subset k[x, y].$$

Then we get a double line in  $\mathbb{A}_k^2$ . Just as in the case of a double point, we can think of the non-reduced structure as being the data of some sort of tangent directions (or better normal directions). Note that

abstractly we have a product

$$\mathbb{A}_k^1 \times T,$$

since

$$k[x, y]/\langle x^2 \rangle \simeq k[y] \otimes k[\epsilon]/\langle \epsilon^2 \rangle.$$

Clearly this structure becomes quite rich if we consider double structures on copies of  $\mathbb{P}^1$ , for example double lines in  $\mathbb{P}^3$ . It turns out that there are continuous non-isomorphic families of double structure on a copy of  $\mathbb{P}^1$ .

It is also interesting to consider embedded components.

**Definition 6.5.** *Let  $X$  be a scheme and let  $Z$  be a locally closed subscheme. The **closure of  $Z$**  is the smallest closed subscheme of  $X$  which contains  $Z$ .*

Of course the closure of  $Z$  is the intersection of all the closed subschemes of  $X$  which contain it. One can also define the closure in terms of the induced immersion

$$Z \longrightarrow X.$$

It is the induced subscheme of  $X$ .

**Definition 6.6.** *If  $X$  is a scheme, then we say that  $X$  has an **embedded component** if there is a dense open subset of  $X$  whose closure is not equal to  $X$ .*

For example, if there is a dense open set  $U$  which is reduced then the closure of  $U$  is reduced, so  $X$  has an embedded component iff  $X$  is not reduced.

In terms of examples, we will only consider non-reduced scheme structures on  $\mathbb{A}_k^1$ . Perhaps the simplest example is to consider the subscheme of  $\mathbb{A}_k^2$  defined by the ideal  $\langle y^2, xy \rangle$ . The support of this closed subscheme is the  $x$ -axis. The open subscheme  $U = U_x$  is a reduced subscheme; on the other hand, this scheme is not reduced as  $y \neq 0 \in k[x, y]/\langle y^2, xy \rangle$ , and  $y^2 = 0$ . Thus the origin is an embedded component.

Note that the ideal of functions vanishing on this scheme is equal to the ideal of functions vanishing along the  $x$ -axis, which also vanish to order two at the origin. Algebraically

$$\langle y^2, xy \rangle = \langle y \rangle \cap \langle x^2, xy, y^2 \rangle.$$

Put differently, the restriction of a function  $f(x, y) \in k[x, y]$  to this scheme is determined by the function  $g(x) = f(x, 0)$  and the value of



the partial derivative

$$\left. \frac{\partial f}{\partial y} \right|_{(0,0)}.$$

Note that it is convenient to think of this scheme as the union of two schemes, the line given by  $\langle y \rangle$  and the 2nd infinitesimal neighbourhood of the origin,

$$\langle x^2, xy, y^2 \rangle.$$

To go further into the theory of embedded components, we need to recall some facts from algebra.

**Definition 6.7.** *Let  $M$  be an  $R$ -module. The **primes associated to  $M$**  are simply the annihilators of any element of  $M$ . The **primes associated to an ideal  $I \subset R$**  are then the primes associated to the quotient  $R/I$ .*

*An ideal  $\mathfrak{q} \subset \mathfrak{p}$  is called **primary to  $\mathfrak{p}$**  if  $\mathfrak{p}$  is the radical of  $\mathfrak{q}$  and for every pair of elements  $f$  and  $g$  of  $R$  if  $fg \in \mathfrak{q}$  and  $f \notin \mathfrak{p}$  then  $g \in \mathfrak{q}$ .*

For example  $\langle x^2, xy, y^2 \rangle$  is primary to  $\langle x, y \rangle$  in the polynomial ring  $k[x, y]$ . Another way to restate the second condition is that the localisation map

$$R/\mathfrak{q} \longrightarrow R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}},$$

is injective.

The key point is that every ideal is the intersection of primary ideals (one should think of this as a factorisation, for example  $\langle 6 \rangle = \langle 2 \rangle \cap \langle 3 \rangle$ ). Unfortunately the elements of the intersection are not unique.

There are ways to eliminate some of the redundancy however. We may assume that no ideal of the intersection can be removed. We may also assume that the primes associated to the primary ideals are distinct (indeed the intersection of two primary ideals with the same prime ideal is primary to this prime ideal). We call this a **primary decomposition** of  $I$  and we call the ideals of the intersection the **primary components** of  $I$ . Now it is true that the set of prime ideals, for which an ideal in the intersection is primary, is unique. In fact these ideals are nothing but the prime ideals associated to  $I$ . The primary ideals of the intersection are not unique. It does turn out that a primary ideal of the intersection is unique, however, if the corresponding prime ideal is minimal.

For example,

$$\langle y^2, xy \rangle = \langle y \rangle \cap \langle x^2, xy, y^2 \rangle,$$

is a primary decomposition of  $I = \langle y^2, xy \rangle$ . The associated primes are  $\langle y \rangle$  and  $\langle x, y \rangle$ . Since  $\langle y \rangle$  is minimal,  $\langle y \rangle$  appears in every primary decomposition of  $I$ . However we could choose  $\langle x, y^2 \rangle$  or  $\langle x + y, y^2 \rangle$  or

indeed  $\langle x + ay, y^2 \rangle$  for the other ideal. Indeed,  $\langle x^n, xy, y^2 \rangle$  will also do, for any  $n \geq 1$ .

Recall the definition of the length.

**Definition 6.8.** *Let  $M$  be an  $R$ -module. The **length** of  $M$  is the maximal length of a chain*

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{l-i} \supset M_l.$$

It turns out that the length of the primary ideal is fixed however; it may be computed as the maximal length of an ideal of finite length in  $R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ .

For example it turns out that for our favourite example, the length is one.